Positivity conjectures for Kazhdan-Lusztig theory on twisted involutions: the universal case

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Abstract

Let (W,S) be any Coxeter system and let $w \mapsto w^*$ be an involution of W which preserves the set of simple generators S. Lusztig has recently shown that the set of twisted involutions (i.e., elements $w \in W$ with $w^{-1} = w^*$) naturally generates a module of the Hecke algebra of (W,S) with two distinguished bases. The transition matrix between these bases defines a family of polynomials $P^\sigma_{y,w}$ which one can view as a "twisted" analog of the much-studied family of Kazhdan-Lusztig polynomials of (W,S). The polynomials $P^\sigma_{y,w}$ can have negative coefficients, but display several conjectural positivity properties of interest. This paper reviews Lusztig's construction and then proves three such positivity properties for Coxeter systems which are universal (i.e., having no braids relations), generalizing previous work of Dyer. Our methods are entirely combinatorial, in contrast to the geometric arguments employed by Lusztig and Vogan to prove similar positivity conjectures for crystallographic Coxeter systems.

Contents

1 Introduction

1.1 Overview

A nice source of open problems in the representation theory of Coxeter systems comes from the frequent observation that interesting properties of Weyl groups seem to hold for much larger classes of reflection groups. This paper concerns phenomena of this nature which have arisen in ongoing work of Lusztig and Vogan [17, 18, 19, 20, 21].

Let (W, S) be any Coxeter system, and write $\mathcal{H}_{q^2}$ for the associated *generic Hecke algebra with parameter* q^2: this is the usual Hecke algebra (i.e., a certain $\mathbb{Z}[q^{\pm 1/2}]$-algebra with a basis $\{T_w : w \in W\}$ indexed by W), but with q replaced by q^2 in its defining relations. A precise definition is given in Section 1.4 below. Next, fix an automorphism $* : W \to W$ which squares to the identity and which preserves the set of simple generators S. Write $\mathbf{I}_*$ for the corresponding set of twisted involutions (i.e., elements $w \in W$ with $w^{-1} = w^*$), and let $\mathcal{M}_{q^2}$ be the free $\mathbb{Z}[q^{\pm 1/2}]$-module which this set generates.

Lusztig [17] has shown that $\mathcal{M}_{q^2}$ has an $\mathcal{H}_{q^2}$-module structure which serves as a natural and interesting analogue of the regular representation of $\mathcal{H}_{q^2}$ on itself. (Skip to Section 1.4 for the details of this construction.) There is a special basis for the regular representation of $\mathcal{H}_{q^2}$ given by the *Kazhdan-Lusztig basis* $(C_w)_{w\in W}$, whose transition matrix from the standard basis $(T_w)_{w\in W}$ defines the much-studied family of *Kazhdan-Lusztig polynomials* $(P_{y,w})_{y,w\in W} \subset \mathbb{Z}[q]$. Lusztig's work [17] indicates that we may repeat much of this theory for the module $\mathcal{M}_{q^2}$: it too has a special "Kazhdan-Lusztig basis" whose transition matrix from the standard basis defines a family of "twisted Kazhdan-Lusztig polynomials" $(P^\sigma_{y,w})_{y,w\in\mathbf{I}_*} \subset \mathbb{Z}[q]$.

The objects in this "twisted Kazhdan-Lusztig theory" display a number of remarkable properties, whose study the present work continues. For example, one of the more famous aspects of the original Kazhdan-Lusztig polynomials $(P_{y,w})_{y,w\in W}$ is that their coefficients appear always to be nonnegative. Kazhdan and Lusztig [12] proved that this holds when W is crystallographic using intersection cohomology some three decades ago, and quite recently, Elias and Williamson [5] have announced an algebraic proof for all Coxeter groups. The twisted Kazhdan-Lusztig polynomials $(P^\sigma_{y,w})_{y,w\in\mathbf{I}_*}$ can have negative coefficients. However, Lusztig and Vogan [19] have shown by geometric arguments that the modified polynomials $\frac{1}{2}(P_{y,w} \pm P^\sigma_{y,w})$ for $y, w \in \mathbf{I}_*$ have nonnegative coefficients whenever W is crystallographic. In fact, for any choice of (W, S) and $*$, the polynomials $\frac{1}{2}(P_{y,w} \pm P^\sigma_{y,w})$ belong to $\mathbb{Z}[q]$, and Lusztig [17] has conjectured that their coefficients are always nonnegative.

Section 1.4 contains two other positivity conjectures for the "Kazhdan-Lusztig basis" of the twisted involution $\mathcal{H}_{q^2}$-module $\mathcal{M}_{q^2}$. These are analogues of longstanding conjectures (soon to be theorems of Elias and Williamson) related to the Kazhdan-Lusztig basis of $\mathcal{H}_{q^2}$, which we review in Section 1.3. Our main results are proofs of the "twisted" conjectures in the case when (W, S) is a universal Coxeter system (i.e., one such that $st \in W$ has infinite order for all distinct $s, t \in S$) and $*$ is arbitrary. This generalizes Dyer's work [4] on the ordinary Kazhdan-Lusztig basis in the universal case. A detailed summary of what is done appears in Section 1.5 at the end of this introduction.

1.2 Setup

Throughout we write $\mathbb{Z}$ for the integers and $\mathbb{N} = \{0, 1, 2, \dots\}$ for the nonnegative integers. We also adopt the following conventions:

- Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$.
- Let $\leq$ denote the Bruhat order on W.
- Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in an indeterminate v.
- Let $q = v^2$. In the sequel, we will refer to v in place of the parameter $q^{1/2}$ used in Section 1.1.

Thus W is a group; $S \subset W$ is a finite set of elements of order two which generate W; $\ell(w)$ is the minimum integer k such that $w = s_1 s_2 \cdots s_k$ for some $s_i \in S$; and we have $y \leq w$ for two elements $y, w \in W$ if and only if whenever $w = s_1 s_2 \cdots s_{\ell(w)}$ for some $s_i \in S$, there are indices $1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq \ell(w)$ such that $y = s_{i_1} s_{i_2} \cdots s_{i_m}$. In particular, if $y < w$ then $\ell(y) < \ell(w)$. Finally, the ring $\mathcal{A}$ will now occupy the role which $\mathbb{Z}[q^{\pm 1/2}]$ played in the previous section. For background on Coxeter systems and the Bruhat order, see for example [1, 10, 16].

1.3 Positivity conjectures from Kazhdan-Lusztig theory

Here we briefly review three longstanding positivity conjectures related to the Hecke algebra of a Coxeter system. These will motivate three analogous conjectures in the next section.

Let $\mathcal{H}_q$ denote the free $\mathcal{A}$-module with basis $\{t_w : w \in W\}$. This module has a unique $\mathcal{A}$-algebra structure with respect to which the multiplication rule

$$t_s t_w = \begin{cases} t_{sw} & \text{if } \ell(sw) = \ell(w) + 1 \\ q t_{sw} + (q-1) t_w & \text{if } \ell(sw) = \ell(w) - 1 \end{cases}$$

holds for each $s \in S$ and $w \in W$.

Remark. The element $t_w \in \mathcal{H}_q$ is more often denoted in the literature by the symbol T_w, but here we reserve the latter notation for the Hecke algebra $\mathcal{H}_{q^2}$, to be introduced in the next section.

We refer to the algebra $\mathcal{H}_q$ as the *Hecke algebra of* (W, S) *with parameter* q. A number of good references exist for this much-studied object; see for example [1, 10, 12, 16]. The Hecke algebra possesses a unique ring involution $\overline{} : \mathcal{H}_q \to \mathcal{H}_q$ with $\overline{v^n} = v^{-n}$ and $\overline{t_w} = (t_{w^{-1}})^{-1}$ all $n \in \mathbb{Z}$ and $w \in W$, referred to as the *bar operator*, and this gives rise to the following well-known theorem-definition.

Theorem-Definition 1.1 (Kazhdan and Lusztig [12])**.** For each $w \in W$ there is a unique family of polynomials $(P_{y,w})_{y \in W} \subset \mathbb{Z}[q]$ with the following three properties:

(a) The element $c_w \overset{\text{def}}{=} v^{-\ell(w)} \cdot \sum_{y \in W} P_{y,w} \cdot t_y \in \mathcal{H}_q$ has $\overline{c_w} = c_w$.

(b) $P_{y,w} = \delta_{y,w}$ if $y \not< w$ in the Bruhat order.

(c) $P_{y,w}$ has degree at most $\frac{1}{2}(\ell(w) - \ell(y) - 1)$ as a polynomial in q whenever $y < w$.

Remark. Here and in the sequel, the Kronecker delta $\delta_{y,w}$ has the usual meaning of $\delta_{y,w} = 1$ if $y = w$ and $\delta_{y,w} = 0$ otherwise.

The polynomials $(P_{y,w})_{y,w \in W}$ are the *Kazhdan-Lusztig polynomials* of the Coxeter system (W, S). Property (b) implies that the elements $(c_w)_{w\in W}$ form an $\mathcal{A}$-basis for $\mathcal{H}_q$, which one calls the *Kazhdan-Lusztig basis*. Many results in the theory of Hecke algebras depend on positivity properties of these objects, and in particular, we have the following conjectures.

Conjecture A. The polynomials $P_{y,w}$ have nonnegative integer coefficients.

Conjecture B. The polynomials $P_{y,w}$ are decreasing for fixed w, in the sense that the difference $P_{y,w} - P_{z,w}$ has nonnegative integer coefficients whenever $y \leq z$.

Denote the structure coefficients of $\mathcal{H}_q$ in the Kazhdan-Lusztig basis by $(h_{x,y;z})_{x,y,z\in W}$; i.e., these are the Laurent polynomials in $\mathcal{A}$ satisfying $c_x c_y = \sum_{z\in W} h_{x,y;z} c_z$ for $x, y, z \in W$.

Conjecture C. The Laurent polynomials $h_{x,y;z}$ have nonnegative coefficients.

These conjectures have been proved in the case when (W, S) is crystallographic (i.e., when W a Weyl group or affine Weyl group), finite, or universal through the work of a number of people [3, 4, 11, 13, 15, 25]. Moreover, as mentioned above, Elias and Williamson [5] have recently announced a forthcoming proof of Soergel's conjecture which affords an algebraic proof of Conjectures A, B, and C for any Coxeter system.

1.4 Positivity conjectures from "twisted Kazhdan-Lusztig theory"

The present work focuses not on Conjectures A, B, and C but rather on a set of analogous conjectures for a certain Hecke algebra module, which we describe in the present section.

Above we defined an $\mathcal{A}$-algebra $\mathcal{H}_q$ possessing two $\mathcal{A}$-bases indexed by W: the standard basis $(t_w)_{w\in W}$ and the Kazhdan-Lusztig basis $(c_w)_{w\in W}$. Here we introduce the slightly different Hecke algebra $\mathcal{H}_{q^2}$, possessing an analogous pair of $\mathcal{A}$-bases indexed by W which we will denote using capital letters by $(T_w)_{w\in W}$ and $(C_w)_{w\in W}$.

In detail, we let $\mathcal{H}_{q^2}$ denote the free $\mathcal{A}$-module with basis $\{T_w : w \in W\}$. This module has a unique $\mathcal{A}$-algebra structure with respect to the slightly altered multiplication rule

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) = \ell(w) + 1 \\ q^2 T_{sw} + (q^2 - 1) T_w & \text{if } \ell(sw) = \ell(w) - 1 \end{cases}$$

holds for each $s \in S$ and $w \in W$. We refer to $\mathcal{H}_{q^2}$ with this stucture as the *Hecke algebra of* (W, S) *with parameter* q^2. This algebra likewise possesses a unique ring involution $\overline{} : \mathcal{H}_{q^2} \to \mathcal{H}_{q^2}$ with $\overline{v^n} = v^{-n}$ and $\overline{T_w} = (T_{w^{-1}})^{-1}$ for all $n \in \mathbb{Z}$ and $w \in W$, which fixes each of the elements

$$C_w \overset{\text{def}}{=} q^{-\ell(w)} \cdot \sum_{y\in W} P_{y,w}(q^2) \cdot T_y \in \mathcal{H}_{q^2} \qquad \text{for } w \in W.$$

The elements $(C_w)_{w\in W}$ form an $\mathcal{A}$-basis of $\mathcal{H}_{q^2}$ which one refers to as the *Kazhdan-Lusztig basis*.

The following Theorem-Definition of Lusztig [17] introduces the $\mathcal{H}_{q^2}$-module will be our central object of study. This statement requires a few additional ingredients:

- Fix an automorphism $w \mapsto w^*$ of W with order ≤ 2 such that $s^* \in S$ for each $s \in S$.
- Set $\mathbf{I}_* = \{w \in W : w^* = w^{-1}\}$. One calls elements of this set *twisted involutions*.
- Given $s \in S$ and $w \in \mathbf{I}_*$, let $s \ltimes w$ denote the unique element in the intersection of $\{sw, sws^*\}$ and $\mathbf{I}_* \setminus \{w\}$. Note that while $s \ltimes (s \ltimes w) = w$, the operation $\ltimes : S \times \mathbf{I}_* \to \mathbf{I}_*$ generally does not extend to a group action of W on $\mathbf{I}_*$.

Lusztig proves the following facts in [17]. This result first appeared in Lusztig and Vogan's paper [19], in the special case that W is a Weyl group or affine Weyl group and $*$ is trivial.

Theorem-Definition 1.2 (Lusztig [17])**.** Let $\mathcal{M}_{q^2}$ be the free $\mathcal{A}$-module with basis $\{a_w : w \in \mathbf{I}_*\}$.

(a) $\mathcal{M}_{q^2}$ has a unique $\mathcal{H}_{q^2}$-module structure with respect to which the multiplication rule

$$T_s a_w = f_{s,w} \cdot a_{s\ltimes w} + g_{s,w} \cdot a_w \tag{1.1}$$

holds for each $s \in S$ and $w \in \mathbf{I}_*$, where $f_{s,w}, g_{s,w} \in \mathbb{Z}[q]$ are the polynomials defined by

$$f_{s,w} - 1 = g_{s,w} = \begin{cases} 0 & \text{if } \ell(s \ltimes w) = \ell(w) + 2 \\ q, & \text{if } \ell(s \ltimes w) = \ell(w) + 1 \\ q^2 - q - 1, & \text{if } \ell(s \ltimes w) = \ell(w) - 1 \\ q^2 - 1, & \text{if } \ell(s \ltimes w) = \ell(w) - 2. \end{cases}$$

(b) There is a unique $\mathbb{Z}$-linear involution $\overline{} : \mathcal{M}_{q^2} \to \mathcal{M}_{q^2}$ such that $\overline{a_1} = a_1$ and $\overline{h \cdot m} = \overline{h} \cdot \overline{m}$ for all $h \in \mathcal{H}_{q^2}$ and $m \in \mathcal{M}_{q^2}$. This bar operator acts on the standard basis of $\mathcal{M}_{q^2}$ by the formula $\overline{a_w} = \mathrm{sgn}(w) \cdot (T_{w^{-1}})^{-1} \cdot a_{w^{-1}}$ for $w \in \mathbf{I}_*$, where $\mathrm{sgn}(w) = (-1)^{\ell(w)}$.

When W is a Weyl group or affine Weyl group, the module $\mathcal{M}_{q^2}$ has a very definite geometric meaning [19]. While for more general Coxeter groups we lack such as interpretation for $\mathcal{M}_{q^2}$, there are still several reasons why this construction is an interesting thing to consider. First, there is a sense in which we can view the left regular representation of the Hecke algebra of a Coxeter system as a special case of (a submodule of) the module $\mathcal{M}_{q^2}$, discussed below in Section 2.3. Moreover, one can show that there are only finitely many formulas of the particular form (1.1) which define, for all choices of (W, S), an $\mathcal{H}_{q^2}$-module structure on $\mathcal{M}_{q^2}$. Up to composition with an automorphism of $\mathcal{H}_{q^2}$, these module structures are either isomorphic to the one given here or a sum of trivial representations, so one can consider the given structure to be in some sense canonical.

We also mention that when W is finite, the irreducible decomposition of $\mathcal{M}_{q^2}$ has a surprising interpretation in terms of the "Fourer transform" of a set of "unipotent characters" attached to (W, S). This phenomenon, which is studied in various cases in the articles [2, 6, 14, 19, 22], gives one more indication that $\mathcal{M}_{q^2}$ deserves consideration not only in the crystallographic case.

The present reason why we study $\mathcal{M}_{q^2}$ derives specifically from the following analogue of Theorem-Definition 1.1. Like the previous result, this was first shown by Lusztig and Vogan [19] in the crystallographic case (with $*$ trivial). Lusztig [17] subsequently extended the statement to all Coxeter systems.

Theorem-Definition 1.3 (Lusztig [17])**.** For each $w \in \mathbf{I}_*$ there is a unique family of polynomials $\left(P^\sigma_{y,w}\right)_{y \in \mathbf{I}_*} \subset \mathbb{Z}[q]$ with the following three properties:

(a) The element $A_w \overset{\text{def}}{=} v^{-\ell(w)} \cdot \sum_{y \in \mathbf{I}_*} P^\sigma_{y,w} \cdot a_y \in \mathcal{M}_{q^2}$ has $\overline{A_w} = A_w$.

(b) $P^\sigma_{y,w} = \delta_{y,w}$ if $y \not< w$ in the Bruhat order.

(c) $P^\sigma_{y,w}$ has degree at most $\frac{1}{2}(\ell(w) - \ell(y) - 1)$ as a polynomial in q whenever $y < w$.

Remark. Note from (b) that the elements $(A_w)_{w \in \mathbf{I}_*}$ form an $\mathcal{A}$-basis for the module $\mathcal{M}_{q^2}$. We sometimes refer to this as the "twisted Kazhdan-Lusztig basis."

The parallels between Theorem-Definitions 1.1 and 1.3 suggest some obvious analogues of the conjectures in the previous section, but these statements turn out not to be the right ones. Notably, the polynomials $P^\sigma_{y,w}$ may have negative coefficients. To state the "correct" conjectures, define $P^+_{y,w}, P^-_{y,w} \in \mathbb{Q}[q]$ by

$$P^\pm_{y,w} = \tfrac{1}{2}\left(P_{y,w} \pm P^\sigma_{y,w}\right) \qquad \text{for each } y, w \in \mathbf{I}_*.$$

Lusztig proves that these polynomials actually have integer coefficients [17, Theorem 9.10] (see also Section 2.4 below), and conjectures the following:

Conjecture A′. The polynomials $P^+_{y,w}$ and $P^-_{y,w}$ have nonnegative integer coefficients.

This statement is a refinement of Conjecture A since $P^+_{y,w} + P^-_{y,w} = P_{y,w}$ for $y, w \in \mathbf{I}_*$. We also introduce the following stronger conjecture, which is likewise a refinement of Conjecture B.

Conjecture B′. The polynomials $P^\pm_{y,w}$ are decreasing for fixed w, in the sense that the differences $P^+_{y,w} - P^+_{z,w}$ and $P^-_{y,w} - P^-_{z,w}$ have nonnegative integer coefficients whenever $y \le z$.

Finally, to provide an analog of Conjecture C, for each $x \in W$ and $y \in \mathbf{I}_*$ define $\left(\widetilde{h}_{x,y;z}\right)_{z \in W} \subset \mathcal{A}$ and $\left(h^\sigma_{x,y;z}\right)_{z \in \mathbf{I}_*} \subset \mathcal{A}$ as the Laurent polynomials satisfying

$$c_x c_y c_{x^{*-1}} = \sum_{z \in W} \widetilde{h}_{x,y;z} c_z \qquad \text{and} \qquad C_x A_y = \sum_{z \in \mathbf{I}_*} h^\sigma_{x,y;z} A_z, \tag{1.2}$$

Note that $c_x, c_y, c_z \in \mathcal{H}_q$ while $C_x \in \mathcal{H}_{q^2}$ and $A_y \in \mathcal{M}_{q^2}$. Now, define $h^+_{x,y;z}, h^-_{x,y;z} \in \mathbb{Q}[v, v^{-1}]$ by

$$h^\pm_{x,y;z} = \tfrac{1}{2}\left(\widetilde{h}_{x,y;z} \pm h^\sigma_{x,y;z}\right), \qquad \text{for each } x \in W \text{ and } y, z \in \mathbf{I}_*. \tag{1.3}$$

One can show from results of Lusztig [17] that these Laurent polynomials likewise have integer coefficients (see Proposition 2.14 below), and we have this conjecture.

Conjecture C′. The Laurent polynomials $h^+_{x,y;z}$ and $h^-_{x,y;z}$ have nonnegative integer coefficients.

1.5 Outline of main results

Lusztig and Vogan's work [17] establishes Conjecture A′ when W is a Weyl group or affine Weyl group by geometric methods. In these cases, [19, Section 5] also mentions (without proof) that Conjecture C′ holds (when $*$ is trivial). Conjecture B′ appears still to be open even in the crystallographic case. Here, we will provide some evidence that these conjectures might hold for all Coxeter systems, by proving them in the case that W is universal.

Following Dyer [4], we say that a Coxeter system (W, S) is *universal* if the product $st \in W$ has infinite order for any distinct generators $s, t \in S$. In this case W is the group generated by S subject only to the relations $s^2 = 1$ for $s \in S$. The elements of W consists of all words in S with distinct adjacent letters, and products of elements are given by concatenation, subject to the rule that one inductively removes all pairs of equal adjacent letters.

Let (W, S) be any universal Coxeter system and let $*$ be any S-preserving involution of W. Restricted to S, the involution then corresponds to either the identity or to an arbitrary permutation of order two. Dyer's paper [4] derives formulas for the polynomials $P_{y,w}$ and for the decomposition of the products $c_x c_y \in \mathcal{H}_q$ in terms of the Kazhdan-Lusztig basis, thus establishing Conjectures A, B, and C in the universal case. (Dyer's results are formulated in somewhat different language than these conjectures; cf. Theorems 3.5 and 3.15 below.) Our paper proceeds as something of a sequel to Dyer's work, as follows:

- In Sections 3.1, 3.2, and 3.3 we derive a series of recurrence relations, with coefficients in $\mathbb{N}[q]$, for the polynomials $P^\sigma_{y,z;w} \overset{\text{def}}{=} P^\sigma_{y,w} - P^\sigma_{z,w}$ and $P_{y,z;w} \overset{\text{def}}{=} P_{y,w} - P_{z,w}$ (with $y \leq z$).
- These recurrences show that in the universal case the polynomials $P^\sigma_{y,w}$ and $P^\pm_{y,w}$ have nonnegative integer coefficients and are decreasing with respect to the index $y \in \mathbf{I}_*$ and the Bruhat order; see Theorems 3.12 and 3.13 below. Thus Conjectures A$'$ and B$'$ hold for universal Coxeter systems.
- In Section 3.4 we describe the decomposition of the product $C_x A_y$ in terms of the distinguished basis $(A_z)_{z \in \mathbf{I}_*}$ of $\mathcal{M}_{q^2}$; see Theorem 3.18. This shows that the Laurent polynomials $h^\sigma_{x,y,;z}$ have nonnegative coefficients in the universal case; see Corollary 3.19.
- Combining these results with Dyer's work finally affords a proof of Conjecture C$'$ for universal Coxeter systems; see Theorem 3.22.

This all is carried out in Section 3. Before this we provide some relevant preliminaries in Section 2. Mainly, we survey a few facts concerning the Bruhat order on $\mathbf{I}_*$, and then review some properties of the polynomials $P_{y,w}$ and $P^\sigma_{y,w}$ and the associated bases of $\mathcal{H}_q$ and $\mathcal{M}_{q^2}$. In particular, Section 2.2 describes an algorithm from [17] for computing the polynomials $P^\sigma_{y,w}$, and Section 2.3 explains the sense in which ordinary Kazhdan-Lusztig theory can be recovered from the twisted case.

Acknowledgements

I am grateful to George Lusztig and David Vogan for some helpful discussions and suggestions.

2 Preliminaries

Here, we preserve all conventions from the introduction. Thus, (W, S) is an arbitrary Coxeter system (not necessarily universal) with an S-preserving involution $* \in \mathrm{Aut}(W)$, and attached to these choices are the following structures:

- $\mathcal{H}_{q^2} = \mathcal{A}\text{-span}\{T_w : w \in W\}$ is the Hecke algebra of (W, S) with parameter q^2.
- $\mathbf{I}_* = \{w \in W : w^{-1} = w^*\}$ is corresponding set of twisted involutions.

- $\mathcal{M}_{q^2} = \mathcal{A}\text{-span}\{a_w : w \in \mathbf{I}_*\}$ is the $\mathcal{H}_{q^2}$-module generated by $\mathbf{I}_*$.

Recall also the definitions of the special bases $(C_w)_{w\in W} \subset \mathcal{H}_{q^2}$ and $(A_w)_{w\in\mathbf{I}_*} \subset \mathcal{M}_{q^2}$, and the polynomials $(P_{y,w})_{y,w\in W}$ and $\left(P^\sigma_{y,w}\right)_{y,w\in\mathbf{I}_*}$ in $\mathbb{Z}[q]$.

2.1 Bruhat order on twisted involutions

The set of twisted involutions $\mathbf{I}_*$ is partially ordered by the Bruhat order $\leq$ on W, and this ordering controls many important features of basis $(A_w)_{w\in\mathbf{I}_*} \subset \mathcal{M}_{q^2}$ and the polynomials $\left(P^\sigma_{y,w}\right)_{y,w\in\mathbf{I}_*} \subset \mathbb{Z}[q]$. The subposet $(\mathbf{I}_*, \leq)$ has a more direct characterization and a number of interesting properties, which are meticulously detailed in Hultman's papers [7, 8, 9]. Hultman's work extends to arbitrary Coxeter systems many earlier observations of Richardson and Springer [23, 24, 26] concerning $\mathbf{I}_*$ when W is finite. Some of this material is useful for our purposes and so we briefly summarize it here.

Recall that we define

$$s \ltimes w \overset{\text{def}}{=} \begin{cases} sw & \text{if } sw = ws^* \\ sws^* & \text{if } sw \neq ws^* \end{cases} \qquad \text{for } s \in S \text{ and } w \in \mathbf{I}_*. \tag{2.1}$$

In [17], Lusztig uses the notation $s \bullet w$ instead of $s \ltimes w$; we prefer the latter notation to emphasize that $s \in S$ acts to "twist" $w \in \mathbf{I}_*$. Although this notation does not extend to an action of W of $\mathbf{I}_*$, it does lead to the following definition, adapted from [7, 8, 9]:

Definition 2.1. A sequence $(s_1, s_2, \dots, s_k)$ with $s_i \in S$ is an $\mathbf{I}_*$-*expression* for a twisted involution $w \in \mathbf{I}_*$ if one of the following occurs:

(i) $k = 0$ and $w = 1$;

(ii) $k > 0$ and $(s_2, \dots, s_k)$ is an $\mathbf{I}_*$-expression for $s \ltimes w$.

Note that these conditions are equivalent to

$$w = s_1 \ltimes (s_2 \ltimes (\cdots \ltimes (s_k \ltimes 1)\cdots)).$$

An $\mathbf{I}_*$-expression for w is *reduced* if its length k is minimal.

What we refer to as $\mathbf{I}_*$-expressions are the left-handed versions of what Hultman terms "$\underline{S}$-expressions" in [7, 8, 9]. (In consequence, all of our statements here are in fact the left-handed versions of Hultman's.) It follows by induction on $\ell(w)$ that every $w \in \mathbf{I}_*$ has a reduced $\mathbf{I}_*$-expression, and so the next statement (given as [9, Proposition 2.5]) is well-defined:

Proposition-Definition 2.2 (Hultman [9])**.** Choose a reduced $\mathbf{I}_*$-expression $(s_1, s_2, \dots, s_k)$ for $w \in \mathbf{I}_*$ and define $w_0 = w$ and $w_i = s_i \ltimes w_{i-1}$ for $1 \leq i \leq k$. Then then number of indices $i \in [k]$ with $s_i w_i = w_i s_i^*$ depends only on w and not on the choice of $\mathbf{I}_*$-expression. Define $\ell^* : \mathbf{I}_* \to \mathbb{N}$ by setting $\ell^*(w)$ equal to this number. (In particular, $\ell^*(1) = 0$ and $\ell(s) = 1$ for any $s \in S \cap \mathbf{I}_*$.)

The function ℓ^* coincides with the map ϕ which Lusztig defines in [17, Proposition 4.5]. This map measures the difference in size between the (ordinary) reduced expressions and reduced $\mathbf{I}_*$-expressions for a twisted involution, in the sense of the following result, which appears as [7, Theorem 4.8].

Theorem-Definition 2.3 (Hultman [7])**.** Let $\rho : \mathbf{I}_* \to \mathbb{N}$ be the map which assigns to $w \in \mathbf{I}_*$ the common length of any of its reduced $\mathbf{I}_*$-expressions. Then the poset $(\mathbf{I}_*, \leq)$ is graded with rank function ρ, and

$$\rho = \tfrac{1}{2}(\ell + \ell^*)\,.$$

In particular, if $w \in \mathbf{I}_*$ and $s \in S$ then $\rho(s \ltimes w) = \rho(w) - 1$ if and only if $\ell(sw) = \ell(w) - 1$.

The notation ρ for the length of a twisted involution's reduced $\mathbf{I}_*$-expressions will prove useful in later sections. We conclude by stating the "subword property" for the Bruhat order on $\mathbf{I}_*$, which appears for arbitrary Coxeter systems as [9, Theorem 2.8].

Theorem 2.4 (Hultman [9])**.** If $y, w \in \mathbf{I}_*$ are twisted involutions, then $y \leq w$ if and only if whenever $(s_1, s_2, \dots, s_k)$ is a reduced $\mathbf{I}_*$-expression for w, there exist indices $1 \leq i_1 < i_2 < \cdots < i_m \leq k$ such that $(s_{i_1}, s_{i_2} \dots, s_{i_m})$ is a reduced $\mathbf{I}_*$-expression for y.

2.2 Twisted Kazhdan-Lusztig theory

While Theorem-Definition 1.3 establishes the existence of the distinguished basis $(A_w)_{w \in \mathbf{I}_*}$ for the $\mathcal{H}_{q^2}$-module $\mathcal{M}_{q^2}$, it gives little indication of how $\mathcal{H}_{q^2}$ acts on this basis, or of how one can compute the polynomials $\left(P^\sigma_{y,w}\right)_{y,w \in \mathbf{I}_*}$. In this section we summarize the main results of Lusztig [17] answering these problems. Such preliminaries will be essential in tackling Conjectures A′, B′, and C′ for universal Coxeter systems.

Notation. We introduce the following notation for the coefficients of $P^\sigma_{y,w}$ of highest possible order. Given $y, w \in \mathbf{I}_*$, let

$$\mu^\sigma(y,w) \overset{\text{def}}{=} \text{the coefficient of } q^{(\ell(w)-\ell(y)-1)/2} \text{ in } P^\sigma_{y,w}$$
$$\nu^\sigma(y,w) \overset{\text{def}}{=} \text{the coefficient of } q^{(\ell(w)-\ell(y)-2)/2} \text{ in } P^\sigma_{y,w}.$$

In turn, for each $s \in S$ define another integer $\mu^\sigma(y, w; s)$ by the following more complicated formula:

$$\mu^\sigma(y,w;s) \overset{\text{def}}{=} \nu^\sigma(y,w) + \delta_{sy,ys^*}\mu^\sigma(sy,w) - \delta_{sw,ws^*}\mu^\sigma(y,sw) - \sum_{x \in \mathbf{I}_*;\, sx<x} \mu^\sigma(y,x)\mu^\sigma(x,w).$$

As usual, the Kronecker delta here means $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise.

Note since $P^\sigma_{y,w}$ is a polynomial in q and not in $v = q^{\frac{1}{2}}$ that $\mu^\sigma(y,w)$ (respectively, $\nu^\sigma(y,w)$) is nonzero only if $y \leq w$ and $\ell(w) - \ell(y)$ is odd (respectively, even). The following observation concerning $\mu^\sigma(y, w; s)$ is helpful to note and requires a short argument. Here and elsewhere, for any $w \in W$ we write

$$\mathrm{Des}_L(w) \overset{\text{def}}{=} \{s \in S : \ell(sw) < \ell(w)\} \qquad \text{and} \qquad \mathrm{Des}_R(w) \overset{\text{def}}{=} \{s \in S : \ell(ws) < \ell(w)\} \tag{2.2}$$

for the corresponding left and right descent sets.

Observation 2.5. Let $y, w \in \mathbf{I}_*$ and $s \in \mathrm{Des}_L(y) \setminus \mathrm{Des}_L(w)$. Then the integer $\mu^\sigma(y, w; s)$ is nonzero only if $\ell(w) - \ell(y)$ is even and $y < s \ltimes w$.

Proof. All terms in the definition of $\mu^\sigma(y,w;s)$ are zero if $\ell(w)-\ell(y)$ is odd. Assume $y \not< s \ltimes w$. Then $y \not< w$ automatically so $\mu^\sigma(y,w;s) = \delta_{sy,ys^*}\mu^\sigma(sy,w)$. This is zero unless $sy = ys^*$, but if $sy = ys^*$ then $sy = s \ltimes y \not< w$, as $s \ltimes y < w$ would imply the contradiction $y < s \ltimes w$ by Theorem 2.4. (In detail, if $s \ltimes y < w$ then adding s to the beginning of any reduced $\mathbf{I}_*$-expression for $s \ltimes y$ or w forms a reduced $\mathbf{I}_*$-expression for y or $s \ltimes w$, respectively.) Thus $\mu^\sigma(s \ltimes y, w) = 0$. $\square$

Finally, define $m^\sigma(y \xrightarrow{s} w) \in \mathcal{A}$ for $y, w \in \mathbf{I}_*$ and $s \in S$ as the Laurent polynomial

$$m^\sigma(y \xrightarrow{s} w) = \mu^\sigma(y,w)(v+v^{-1}) + \mu^\sigma(y,w;s). \tag{2.3}$$

Observe that $m^\sigma(y \xrightarrow{s} w) = \mu^\sigma(y,w)(v+v^{-1})$ if $\ell(w)-\ell(y)$ is odd and $m^\sigma(y \xrightarrow{s} w) = \mu^\sigma(y,w;s)$ if $\ell(w)-\ell(y)$ is even. Lusztig proves the following result, which explains our notation, as [17, Theorem 6.3].

Theorem 2.6 (Lusztig [17])**.** Let $w \in \mathbf{I}_*$ and $s \in S$. Then $C_s = q^{-1}(T_s+1)$ and

$$C_s A_w = \begin{cases} \left(q+q^{-1}\right) A_w & \text{if } s \in \mathrm{Des}_L(w) \\ \left(v+v^{-1}\right) A_{sw} + \sum_{y \in \mathbf{I}_*;\, sy<y<sw} m^\sigma(y \xrightarrow{s} w) A_y & \text{if } s \notin \mathrm{Des}_L(w) \text{ and } sw = ws^* \\ A_{sws^*} + \sum_{y \in \mathbf{I}_*;\, sy<y<sws^*} m^\sigma(y \xrightarrow{s} w) A_y & \text{if } s \notin \mathrm{Des}_L(w) \text{ and } sw \neq ws^*. \end{cases}$$

Comparing coefficients of a_y on both sides of the preceding equation yields a recurrence for the polynomials $P^\sigma_{y,w}$. Rewriting the right hand side in the standard basis $(a_w)_{w\in\mathbf{I}_*}$ is straightforward from the definitions in Section 1.4, while rewriting the left hand side can be done using the identities $C_s = q^{-1}(T_s+1)$ and $A_w = v^{-\ell(w)}\sum_{y\in\mathbf{I}_*} P^\sigma_{y,w} a_y$ with the multiplication rule (1.1). In this manner one obtains the following corollary:

Corollary 2.7. Let $y, w \in \mathbf{I}_*$ with $y \leq w$ and $s \in \mathrm{Des}_L(w)$.

(a) $P^\sigma_{y,w} = P^\sigma_{s\ltimes y,w}$.

(b) If $s \in \mathrm{Des}_L(y)$ and we let $w' = s \ltimes w$ and $c = \delta_{sw,ws^*}$ and $d = \delta_{sy,ys^*}$, then

$$(q+1)^c P^\sigma_{y,w} = (q+1)^d P^\sigma_{s\ltimes y,w'} + q(q-d)P^\sigma_{y,w'} - \sum_{\substack{z\in\mathbf{I}_*;\, sz<z \\ y\leq z<w}} v^{\ell(w)-\ell(z)+c} \cdot m^\sigma(z \xrightarrow{s} w') \cdot P^\sigma_{y,z}. \tag{2.4}$$

Remark. Because of the $(q+1)^c$ factor on the left, it is not obvious from the recurrence in part (b) that $P^\sigma_{y,w} \in \mathcal{A}$ is actually a polynomial in q (but it is clear that $P^\sigma_{y,w}$ is a rational function of q). That $P^\sigma_{y,w} \in \mathbb{Z}[q]$ follows from results in [17], however, and in light of this, the recurrence shows that $(q+1)^{\ell^*(w)-\ell^*(y)}$ divides $P^\sigma_{y,w}$. Recall here the definition of ℓ^* from Proposition-Definition 2.2.

Translating the preceding result into an algorithm for computing the polynomials $P^\sigma_{y,w}$ involves a little subtlety, because terms on the right hand side of (2.4) can depend on $P^\sigma_{y,w}$. (There is actually only one such term: the summand indexed by $z = y$ when $sw = ws^*$.) Some consideration shows that the following procedure will work to compute the polynomials $P^\sigma_{y,w}$ for all elements $y, w \in \mathbf{I}_*$ up to any given rank in the poset $(\mathbf{I}_*, \leq)$. Lusztig and Vogan describe a similar algorithm in [19, Section 4.5] in the case that (W,S) is crystallographic and $* = 1$ (but their statements actually hold more generally by results in [17].)

Algorithm 2.8. Fix elements $y, w \in \mathbf{I}_*$.

- If $y \not\leq w$ then $P^\sigma_{y,w} = 0$.
- If $y = w$ then $P^\sigma_{y,w} = 1$.

If neither of these cases occurs then $y < w$ so there exists $s \in \mathrm{Des}_L(w)$. By induction we may assume known all polynomials $P^\sigma_{y',w'}$ with either $w' < w$ or $y < y' \leq w' = w$.

- If $s \notin \mathrm{Des}_L(y)$ then $P^\sigma_{y,w} = P^\sigma_{s\ltimes y,w}$ which has already been computed.
- If $s \in \mathrm{Des}_L(y)$ and either $sw \neq ws^*$ or $\ell(w) - \ell(y)$ is even, then the right hand side of (2.4) depends only on polynomials $P^\sigma_{y',w'}$ which have already been computed, and so (2.4) gives a computable formula for $P^\sigma_{y,w}$.
- If $s \in \mathrm{Des}_L(y)$ and $sw = ws^*$ and $\ell(w) - \ell(y)$ is odd, then (2.4) reduces to the form

  $$(q+1)P^\sigma_{y,w} = f + q^{(\ell(w)-\ell(y)+1)/2}\mu^\sigma(y,w)$$

  where $f \in \mathbb{Z}[q]$ is determined by known polynomials $P^\sigma_{y',w'}$. To compute $P^\sigma_{y,w}$ it suffices to compute f, since if we set $n = \frac{1}{2}(\ell(w) - \ell(y) - 1)$ and write $P^\sigma_{y,w} = a_0 + a_1 q + \cdots + a_n q^n$ then

  $$f = a_0 + (a_0 + a_1)q + (a_1 + a_2)q^2 + \cdots + (a_{n-1} + a_n)q^n.$$

  One computes the polynomial f by just evaluating the right hand side of (2.4) under the assumption that $P^\sigma_{y,w} = 0$. (All terms on the right hand side of (2.4) depend on either known polynomials or the unknown polynomial $P^\sigma_{y,w}$, so assuming $P^\sigma_{y,w} = 0$ makes the right hand side of (2.4) computable, and what you get from this computation is f.)

We mention two additional properties of the polynomials $P^\sigma_{y,w}$. Both follow from Corollary 2.7 in a straightforward manner by induction on $\ell(w)$. Lusztig states part (b) explicitly as [17, Proposition 4.10], but this actually serves in [17] as a preliminary to the other results given here.

Corollary 2.9 (Lusztig [17])**.** Let $y, w \in \mathbf{I}_*$ with $y \leq w$.

(a) $P^\sigma_{y,w} = P^\sigma_{y^{-1},w^{-1}} = P^\sigma_{y^*,w^*}$.

(b) $P^\sigma_{y,w}$ has constant coefficient 1.

2.3 Recovering ordinary Kazhdan-Lusztig theory from the twisted case

Here we explain how the Kazhdan-Lusztig polynomials $P_{y,w}$ can occur as instances of the twisted polynomials $P^\sigma_{y,w}$ for certain choices of (W,S) and $*$. This makes it possible to recover several important properties of the Kazhdan-Lusztig basis $(c_w)_{w\in W} \subset \mathcal{H}_q$ and the Kazhdan-Lusztig polynomials $(P_{y,w})_{w\in W} \subset \mathbb{Z}[q]$ from analogous statements for $\mathcal{M}_{q^2}$ in the previous section.

We begin with the following observation. In approaching this statement it is helpful to recall the definitions of the Laurent polynomials $P_{y,w}$, $P^\sigma_{y,w}$, $h_{x,y;z}$, $\widetilde{h}_{x,y;z}$, $h^\sigma_{x,y;z}$ from Sections 1.3 and 1.4.

Observation 2.10. Suppose that (W,S) has the form

$$W = W' \times W' \qquad \text{and} \qquad S = \{(s,1) : s \in S'\} \cup \{(1,s) : s \in S'\} \tag{2.5}$$

for some Coxeter system (W', S'), and that $* \in \mathrm{Aut}(W)$ acts by $(x,y)^* = (y,x)$ for $y, w \in W'$. Let $\mathcal{H}'_{q^2}$ denote the Hecke algebra of (W', S') with parameter q^2.

(a) The map $w \mapsto (w, w^{-1})$ defines a poset isomorphism $(W', \leq) \xrightarrow{\sim} (\mathbf{I}_*, \leq)$.

(b) The unique $\mathcal{A}$-linear map $\mathcal{H}'_{q^2} \to \mathcal{H}_{q^2}$ with $T_w \mapsto T_{(w,1)}$ for $w \in W'$ is an injective algebra homomorphism. With respect to this embedding, the unique $\mathcal{A}$-linear map $\mathcal{H}'_{q^2} \to \mathcal{M}_{q^2}$ with $T_w \mapsto a_{(w,w^{-1})}$ for $w \in W'$ is an isomorphism of left $\mathcal{H}'_{q^2}$-modules. This map commutes with the bar operators of $\mathcal{H}'_{q^2}$ and $\mathcal{M}_{q^2}$ in the sense that $\overline{T_w} \mapsto \overline{a_{(w,w^{-1})}}$ for $w \in W'$.

(c) For all $y, w \in W'$, we have

$$P_{(y,y^{-1}),(w,w^{-1})} = (P_{y,w})^2 \qquad \text{and} \qquad P^\sigma_{(y,y^{-1}),(w,w^{-1})} = P_{y,w}(q^2).$$

(d) For all $x, y, z \in W'$, we have

$$\widetilde{h}_{(x,x^{-1}),(y,y^{-1});(z,z^{-1})} = (\widetilde{h}_{x,y;z})^2 \qquad \text{and} \qquad h^\sigma_{(x,x^{-1}),(y,y^{-1});(z,z^{-1})} = h_{x,y;z}(v^2).$$

Remark. In the four identities in parts (c) and (d), the left expressions are (Laurent) polynomials attached to the Coxeter system (W, S), while the right expressions are quantities defined in terms of the (Laurent) polynomials attached to the Coxeter system (W', S'). In particular, $h_{x,y;z}(v^2)$ denotes the Laurent polynomial obtained by replacing the parameter v with $q = v^2$ in $h_{x,y;z} \in \mathcal{A}$.

Proof. Part (a) is straightforward and has been noted previously as [8, Example 3.2] and also as [23, Example 10.1]. Since $s \ltimes w = sws^*$ for all $s \in S$ and $w \in W$, part (a) implies that the map in part (b) is an $\mathcal{A}$-linear bijection satisfying

$$T_x T_w \mapsto T_{(x,1)} a_{(w,w^{-1})} \qquad \text{for } x, w \in W'.$$

Thus our map is a left $\mathcal{H}'_{q^2}$-module isomorphism with respect to the embedding $\mathcal{H}'_{q^2} \hookrightarrow \mathcal{H}_{q^2}$. To see that this map commutes with the bar operators, note that if $w \in W'$ then $T_w = T_w \cdot T_1$ and $a_{(w,w^{-1})} = T_{(w,1)} \cdot a_1$, so

$$\overline{T_w} = T^{-1}_{w^{-1}} \cdot T_1 \mapsto T^{-1}_{(w^{-1},1)} \cdot a_1 = \overline{a_{(w,w^{-1})}}.$$

The remaining parts follow as an easy exercise from part (b) and the defining properties of $P_{y,w}$ and $P^\sigma_{y,w}$ in Theorem-Definitions 1.1 and 1.2. $\square$

To see an application this observation, define for $y, w \in W$ the integer

$$\mu(y, w) \stackrel{\text{def}}{=} \text{the coefficient of } q^{(\ell(w)-\ell(y)-1)/2} \text{ in } P_{y,w}.$$

As with $\mu^\sigma(y, w)$, the integer $\mu(y, w)$ can only be nonzero if $y \leq w$ and $\ell(w) - \ell(y)$ is odd. If we assume the hypotheses of Observation 2.10, then $sw \neq ws*$ for all $s \in S$ and $w \in \mathbf{I}_*$ and every twisted involution in $\mathbf{I}_*$ has even length. In consequence, if $s \in S'$ and $y, w \in W'$ then it follows by part (c) of the proposition that

$$m^\sigma\left((y, y^{-1}) \xrightarrow{(s,1)} (w, w^{-1})\right) = \nu^\sigma\left((y, y^{-1}), (w, w^{-1})\right) = \mu(y, w).$$

If we substitute this formula into Theorem 2.6 and Corollary 2.7 (and then replace the Coxeter system (W', S') with (W, S) and the parameter q with v) then we obtain the following "untwisted" analogue of Theorem 2.6:

Theorem 2.11 (Kazhdan and Lusztig [12])**.** Let $w \in W$ and $s \in S$. Then $c_s = v^{-1}(t_s + 1)$ and

$$c_s c_w = \begin{cases} (v+v^{-1})c_w & \text{if } s \in \mathrm{Des}_L(w) \\ c_{sw} + \sum_{z\in W;\, sz<z<w} \mu(z,w) c_z & \text{if } s \notin \mathrm{Des}_L(w). \end{cases}$$

There are likewise analogues of Corollaries 2.7 and 2.9. The only part of these results (which are well-known; see for example [10, Chapter 7]) we will need later are the following facts:

Corollary 2.12. Let $y, w \in W$.

(a) If $s \in \mathrm{Des}_L(w)$ then $P_{y,w} = P_{sy,w}$.

(b) $P_{y,w} = P_{y^{-1},w^{-1}} = P_{y^*,w^*}$.

Remark. Since $\mathrm{Des}_L(w) = \mathrm{Des}_R(w^{-1})$, we have also that if $s \in \mathrm{Des}_R(w)$ then $P_{y,w} = P_{ys,w}$.

2.4 Parity statements

Conjectures A$'$, B$'$ and C$'$ are statements concerning whether the Laurent polynomials

$$P^{\pm}_{y,w} = \tfrac{1}{2}\left(P_{y,w} \pm P^{\sigma}_{y,w}\right) \qquad \text{and} \qquad h^{\pm}_{x,y;z} = \tfrac{1}{2}\left(\widetilde{h}_{x,y;z} \pm h^{\sigma}_{x,y;z}\right)$$

for $x \in W$ and $y, z, w \in \mathbf{I}_*$ have nonnegative integer coefficients. It is not clear *a priori* that these polynomials even have integer coefficients, and we spend this last preliminary section clarifying this property.

Here we write $f \equiv g \pmod 2$ for two Laurent polynomials $f, g \in \mathcal{A}$ if $f - g$ has only even integer coefficients; i.e., if $f - g = 2h$ for some $h \in \mathcal{A}$. Lusztig proves the following result, showing that $P^{\pm}_{y,w} \in \mathbb{Z}[q]$, as [17, Theorem 9.10].

Proposition 2.13 (Lusztig [17])**.** For all $y, w \in \mathbf{I}_*$ we have $P_{y,w} \equiv P^{\sigma}_{y,w} \pmod 2$.

Recall the definitions of the Laurent polynomials $\widetilde{h}_{x,y;z}$ and $h^{\sigma}_{x,y;z}$ from (1.2). The following proposition shows that $h^{\pm}_{x,y;z} \in \mathbb{Z}[v, v^{-1}]$. In the case that (W, S) is a Weyl group or affine Weyl group and $*$ is trivial, this property was mentioned without proof in [19, Section 5].

Proposition 2.14. For all $x \in W$ and $y, z \in \mathbf{I}_*$ we have $\widetilde{h}_{x,y,z} \equiv h^{\sigma}_{x,y,z} \pmod 2$.

Proof. Recall that we denote the bases of $\mathcal{H}_q = \mathcal{A}\text{-span}\{t_w : w \in W\} = \mathcal{A}\text{-span}\{c_w : w \in W\}$ using lower case letters and the bases of $\mathcal{H}_{q^2} = \mathcal{A}\text{-span}\{T_w : w \in W\} = \mathcal{A}\text{-span}\{C_w : w \in W\}$ using upper case letters. To abbreviate, we let $w^\dagger = w^{*-1}$ for $w \in W$ and define $h \mapsto h^\dagger$ as the unique $\mathcal{A}$-algebra anti-automorphism of $\mathcal{H}_q$ such that $(t_w)^\dagger = t_{w^\dagger}$. Finally write $\mathrm{proj} : \mathcal{H}_q \to \mathcal{M}_{q^2}$ for the intuitive projection $\sum_{w\in W} p_w t_w \mapsto \sum_{w \in \mathbf{I}_*} p_w a_w$ for any polynomials $p_w \in \mathcal{A}$.

We write $m \equiv m' \pmod 2$ for two elements $m, m' \in \mathcal{M}_{q^2}$ if $m - m' = 2m''$ for some $m'' \in \mathcal{M}_{q^2}$. With respect to this notation, Lusztig [17, 9.4(a)] proves that

$$\mathrm{proj}\left(t_x t_y (t_x)^\dagger\right) \equiv T_x a_y \pmod 2 \qquad \text{for all } x \in W \text{ and } y \in \mathbf{I}_*. \tag{2.6}$$

The current proposition derives from this fact in the following way. Let $x \in W$ and $y \in \mathbf{I}_*$ and note that $(c_w)^\dagger = c_{w^\dagger}$ by Corollary 2.12. The anti-automorphism $\dagger$ consequently preserves

$c_x c_y c_{x^\dagger}$, so we must have $\widetilde{h}_{x,y,z} = \widetilde{h}_{x,y,z^\dagger}$ for all $z \in W$ and it follows that we can write $c_x c_y c_{x^\dagger} = (a + a^\dagger) + \sum_{z \in \mathbf{I}_*} \widetilde{h}_{x,y,z} c_z$ for an element $a \in \mathcal{H}_q$. Since $\mathrm{proj}(a + a^\dagger) \equiv 0 \pmod 2$ and since $\mathrm{proj}(c_z) \equiv A_z \pmod 2$ for $z \in \mathbf{I}_*$ by Proposition 2.13, we deduce that

$$\mathrm{proj}(c_x c_y c_{x^\dagger}) \equiv \sum_{z \in \mathbf{I}_*} \widetilde{h}_{x,y,z} A_z \pmod 2. \tag{2.7}$$

On the other hand, by definition

$$c_x c_y c_{x^\dagger} = v^{-2\ell(x) - \ell(y)} \sum_{x', x'', z \in W} P_{x',x} P_{x'',x} P_{z,y} \cdot t_{x'} t_z (t_{x''})^\dagger.$$

Since $P_{z,y} = P_{z^\dagger,y}$ for all $z \in W$ as $y = y^\dagger$, the anti-automorphism $\dagger$ acts on the right hand sum by exchanging the summands indexed by (x', x'', z) and $(x'', x', z^\dagger)$. It follows by dividing this sum into two parts, consisting of the summands fixed and unfixed by $\dagger$, that we can write

$$c_x c_y c_{x^\dagger} = (b + b^\dagger) + v^{-2\ell(x) - \ell(y)} \sum_{x' \in W} \sum_{z \in \mathbf{I}_*} (P_{x',x})^2 P_{z,y} \cdot t_{x'} t_z (t_{x'})^\dagger.$$

for another element $b \in \mathcal{H}_q$. Since $\mathrm{proj}(b + b^\dagger) \equiv 0 \pmod 2$ and $(P_{x',x})^2 \equiv P_{x',x}(q^2) \pmod 2$ and $P_{z,y} \equiv P^\sigma_{z,y} \pmod 2$ for $y, z \in \mathbf{I}_*$, applying (2.6) to the preceding equation shows that

$$\mathrm{proj}\,(c_x c_y c_{x^\dagger}) \equiv C_x A_y \pmod 2 \qquad \text{for all } x \in W \text{ and } y \in \mathbf{I}_*. \tag{2.8}$$

The proposition now follows immediately by combining (2.7) and (2.8). $\square$

3 Positivity results for universal Coxeter systems

In this section we prove our main results. Thus, for the duration we let (W, S) denote a fixed universal Coxeter system and we let $*$ denote a fixed S-preserving involution of W. It is helpful to recall that the involution $*$ of W corresponds to an arbitrary choice of a permutation with order ≤ 2 of the set S. The twisted involutions $w \in \mathbf{I}_* = \{x \in W : x^{-1} = x^*\}$ each take one of two possible forms:

- If $\ell(w)$ is even then $w = x^* x^{-1}$ for some $x \in W$.
- If $\ell(w)$ is odd then $w = x^* s x^{-1}$ for some $x \in W$ and $s \in S$ with $s = s^*$.

The following observation enumerates a few other special properties of universal Coxeter systems which make them tractable test cases for general questions and conjectures.

Observation 3.1. Assume (W, S) is a universal Coxeter system.

(a) Each $w \in W$ has a unique reduced expression.

(b) Each $w \in \mathbf{I}_*$ has a unique reduced $\mathbf{I}_*$-expression.

(c) If $w \in W \setminus \{1\}$ then $|\mathrm{Des}_L(w)| = |\mathrm{Des}_R(w)| = 1$.

(d) The map $S \times \mathbf{I}_* \to \mathbf{I}_*$ given by $(s, w) \mapsto s \ltimes w$, with $s \ltimes w$ defined by (2.1), extends to a group action of W on $\mathbf{I}_*$.

Notation. In light of part (d), it is well-defined to set

$$x \ltimes w \overset{\text{def}}{=} s_1 \ltimes (s_2 \ltimes (\cdots \ltimes (s_n \ltimes w)\cdots))$$

where $x \in W$ and $w \in \mathbf{I}_*$ and $s_i \in S$ are such that $x = s_1 s_2 \cdots s_n$.

Before proceeding, we note as a second consequence of our observation that one can view the ordinary Kazhdan-Lusztig theory of a universal Coxeter system as a special case of its twisted theory (in a slightly different sense than Observation 2.10). More precisely, if the involution $*$ has no fixed points in S, then the ordinary Kazhdan-Lusztig polynomials $(P_{y,w})_{y,w\in W}$ of a universal Coxeter group coincide with the twisted polynomials $\left(P^\sigma_{y,w}\right)_{y,w\in\mathbf{I}_*}$ in the following way:

Observation 3.2. Suppose (W,S) is a universal Coxeter system and $s \neq s^*$ for all $s \in S$. Then the unique $\mathcal{A}$-linear map $\mathcal{H}_{q^2} \to \mathcal{M}_{q^2}$ with $T_{w^*} \mapsto a_{w^*w^{-1}}$ for $w \in W$ defines an isomorphism of left $\mathcal{H}_{q^2}$-modules which commutes with the bar operators of $\mathcal{H}_{q^2}$ and $\mathcal{M}_{q^2}$, and consequently

$$P^\sigma_{(y^*y^{-1}),(w^*w^{-1})} = P_{y,w}(q^2) \quad\text{and}\quad h^\sigma_{x,\,(y^*y^{-1});\,(z^*z^{-1})} = h_{x,y^*;z^*}(v^2) \qquad \text{for all } w,x,y,z \in W.$$

Proof. If $s \neq s^*$ for all $s \in S$ then every twisted involution has even length and the map $w^* \mapsto w^*w^{-1}$ defines a poset isomorphism $(W,\leq) \xrightarrow{\sim} (\mathbf{I}_*,\leq)$. From this, the proof of the proposition is similar to that of Observation 2.10. ☐

3.1 Kazhdan-Lusztig polynomials

Dyer has derived a formula for the Kazhdan-Lusztig polynomials of a universal Coxeter system [4, Theorem 3.8], showing their coefficients to be nonnegative. We review the key parts of this result here. To begin, we recall the following special case of [4, Lemma 3.5].

Lemma 3.3 (Dyer [4])**.** Assume (W,S) is a universal Coxeter system. Suppose $y,w \in W$ and $r,s \in S$ such that $rsw < sw < w$ and $sy > y$. Then

$$P_{y,w} = P_{y,sw} + qP_{sy,sw} - \delta \cdot qP_{y,rsw} \qquad \text{where } \delta = |\{s\} \cap \mathrm{Des}_L(rsw)|.$$

In the sequel we adopt the following notation. Given $y,z,w \in W$ with $y \leq z$, define

$$P_{y,z;w} = P_{y,w} - P_{z,w}. \tag{3.1}$$

We expand upon the previous lemma with the following statement.

Proposition 3.4. Assume (W,S) is a universal Coxeter system. Let $y,z \in W$ with $y \leq z$ and suppose

- k is a positive integer;
- $r,s \in S$ such that $r \neq s$ and $s \notin \mathrm{Des}_L(y)$ and $s \notin \mathrm{Des}_L(z)$;
- $u \in W$ such that $\{r,s\} \cap \mathrm{Des}_L(u) = \varnothing$.

If $a,w \in W$ are defined by

$$w = \underbrace{srsrs\cdots}_{k+1\text{ factors}} u \qquad\text{and}\qquad a = \underbrace{\cdots srsrs}_{k\text{ factors}}$$

then $P_{y,z;w} = P_{y,z;sw} + q^k P_{ay,az;aw}$.

Remark. Applying the identity $P_{y,z;w} = P_{y^{*-1},z^{*-1};w^{*-1}}$ from Corollary 2.12 affords a right-handed version of this proposition, which will be of use in Section 3.3 below.

Proof. Note that $y \leq z$ implies $ay \leq az$, since (as $sy > y$ and $sz > z$) the unique reduced expression for ay (respectively, az) is formed by concatenating $\cdots srsrs$ to the unique reduced expression for y (respectively, z). To prove the lemma, we proceed by induction on k. If $k = 1$ then the lemma reduces to Lemma 3.3. If $k > 1$, then since $P_{sy,sz;rsw} = P_{y,z;rsw}$ by Corollary 2.12, Lemma 3.3 asserts that $P_{y,z;w} = P_{y,z;sw} + q(P_{sy,sz;sw} - P_{sy,sz;rsw})$. By induction we may assume that $P_{sy,sz;sw} = P_{sy,sz;rsw} + q^{k-1}P_{ay,az;aw}$; substituting this identity into the preceding equation gives the desired recurrence. $\square$

From the last lemma we have an easy proof of Conjecture B (and so also of Conjecture A) for universal Coxeter systems. This result can also be deduced from [4, Theorem 3.8].

Theorem 3.5 (Dyer [4])**.** If (W, S) is a universal Coxeter system, then the polynomial $P_{y,w} - P_{z,w}$ has nonnegative integer coefficients for all $y, z, w \in W$ with $y \leq z$ in the Bruhat order. In particular, we have $P_{y,w} \in \mathbb{N}[q]$ for each $y, w \in W$.

Proof. The proof is by induction on $\ell(w)$. If $\ell(w) \leq 1$ then $P_{y,z;w} \in \{0, 1\} \subset \mathbb{N}[q]$ by Corollary 2.12. Assume $\ell(w) \geq 2$ so that there exists $s \in \mathrm{Des}_L(w)$. Let (y', z') be the unique pair in the set $\{(y, z), (sy, z), (y, sz), (sy, sz)\}$ which has $s \notin \mathrm{Des}_L(y')$ and $s \notin \mathrm{Des}_L(z')$. It is straightforward to check that $y' \leq z'$, and by Corollary 2.12 we have $P_{y,z;w} = P_{y',z';w}$. Proposition 3.4 applied $P_{y',z';w}$ shows that that $P_{y,z;w} \in \mathbb{N}[q]$ by induction, and it follows that $P_{y,w} \in \mathbb{N}[q]$ since $P_{w,w} = 1$. $\square$

3.2 Twisted Kazhdan-Lusztig polynomials

Here we initiate the proof of Conjecture B′ for universal Coxeter systems, to be completed in the next section. As above, (W, S) is a fixed Universal Coxeter system with a fixed S-preserving involution $*$.

Recall the definition of the Laurent polynomial $m^\sigma(y \xrightarrow{s} w) \in \mathcal{A}$ from (2.3).

Lemma 3.6. Assume (W, S) is a universal Coxeter system. If $y, w \in \mathbf{I}_*$ and $r, s \in S$ such that $\mathrm{Des}_L(y) = \{s\} \neq \{r\} = \mathrm{Des}_L(w)$, then

$$m^\sigma(y \xrightarrow{s} w) = \begin{cases} 1 & \text{if } y = rwr^* \text{ or if } (y, w) = (s, r) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First note that since W is a universal Coxeter group and $y \notin \{1, r\}$ we have $r \ltimes y = ryr^*$ and $\ell(r \ltimes y) = \ell(y) + 2$. In addition, from Corollary 2.7 we have $P^\sigma_{y,w} = P^\sigma_{r\ltimes y,w}$.

We claim that $\mu^\sigma(y, w) = 0$. To prove this, note that if $r \ltimes y = w$ then $\ell(w) - \ell(y)$ is even, and if $r \ltimes y \not\leq w$ then $y \not< w$, so in either case $\mu^\sigma(y, w) = 0$. On other other hand, if $r \ltimes y < w$ then the degree of $P^\sigma_{y,w} = P^\sigma_{r\ltimes y,w}$ as a polynomial in q is at most $\frac{\ell(w)-\ell(r\ltimes y)-1}{2}$ which is strictly less than $\frac{\ell(w)-\ell(y)-1}{2}$, so again $\mu^\sigma(y, w) = 0$.

It thus suffices to show that $\mu^\sigma(y, w; s) = 1$ if $y = rwr^*$ or if $(y, w) = (s, r)$ and $\mu^\sigma(y, w; s) = 0$ otherwise. To this end, observe that if we apply our first claim to the definition of $\mu^\sigma(y, w; s)$, and also note that $sw \neq ws^*$ since $w \notin \{1, s\}$, we obtain

$$\mu^\sigma(y, w; s) = \nu^\sigma(y, w) + \delta_{sy,ys^*}\mu^\sigma(sy, w).$$

If $y = rwr^*$ then $P^\sigma_{y,w} = P^\sigma_{w,w} = 1$ so $\nu^\sigma(y,w) = 1$. Alternatively, if $y < w$ and $y \neq rwr^*$ then it follows as above that $P^\sigma_{y,w} = P^\sigma_{r\ltimes y,w}$ has degree strictly less than $\frac{\ell(w)-\ell(y)-2}{2}$ so $\nu^\sigma(y,w) = 0$. Hence

$$\nu^\sigma(y,w) = \begin{cases} 1 & \text{if } y = rwr^* \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

In turn, we have $sy = ys^*$ if and only if $y = s$ (note that $y \neq 1$ by hypothesis), in which case $\mu^\sigma(sy,w) = \mu^\sigma(1,w)$. If $w = r$ then $\mu^\sigma(1,w) = 1$ and if $w \neq r$ then either $w = rr^*$ (in which case $\ell(w) - \ell(1)$ is even) or $r \ltimes 1 \neq w$ (in which case $P^\sigma_{1,w} = P^\sigma_{r\ltimes 1,w}$ has degree strictly less than $\frac{\ell(w)-\ell(1)-1}{2}$) so $\mu^\sigma(1,w) = 0$. Thus

$$\delta_{sy,ys^*}\nu^\sigma(sy,w) = \begin{cases} 1 & \text{if } (y,w) = (s,r) \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Combining (3.2) and (3.3) gives the desired formula for $\mu^\sigma(y,w;s)$. $\square$

We now have the following analogue of Lemma 3.3.

Lemma 3.7. Assume (W,S) is a universal Coxeter system. Suppose $y,w \in \mathbf{I}_*$ and $r,s \in S$ such that $rs \ltimes w < s \ltimes w < w$ and $sy > y$. Then

$$P^\sigma_{y,w} = P^\sigma_{y,s\ltimes w} + q^2 P^\sigma_{s\ltimes y,s\ltimes w} - \delta \cdot q^2 P^\sigma_{s\ltimes y,rs\ltimes w}. + \delta' \cdot q(P^\sigma_{1,s\ltimes w} - P^\sigma_{s,s\ltimes w}).$$

where we define

$$\delta = \begin{cases} 1 & \text{if } s \in \mathrm{Des}_L(rs \ltimes w) \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \delta' = \begin{cases} 1 & \text{if } y = 1 \text{ and } s = s^* \text{ and } w \neq srs \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Everything follows by combining Lemma 3.7 with Corollary 2.7. It is straightforward to check that the lemma holds if $y \not< w$, so assume $y < w$. Let $\delta'' = \delta_{sy,ys^*}$ and note by hypothesis that $sw \neq ws^*$. By Corollary 2.7 we therefore have

$$P^\sigma_{y,w} = P^\sigma_{y,s\ltimes w} + q^2 P^\sigma_{s\ltimes y,s\ltimes w} + \delta'' q(P^\sigma_{y,s\ltimes w} - P^\sigma_{s\ltimes y,s\ltimes w}) - \sum_{\substack{z\in\mathbf{I}_*;\, sz<z \\ y\leq z<w}} v^{\ell(w)-\ell(z)} m^\sigma(z \xrightarrow{s} s\ltimes w) P^\sigma_{y,z}. \tag{3.4}$$

From the preceding lemma we know that $m^\sigma(z \xrightarrow{s} s \ltimes w) = 1$ if $z = rs \ltimes w$ or $(z, s\ltimes w) = (s,r)$, and $m^\sigma(z \xrightarrow{s} s\ltimes w) = 0$ otherwise. The sum in (3.4) includes a summand indexed by $z = rs \ltimes w$ if and only if $\delta = 1$. On the other hand, if $s \ltimes w = r$ then the sum includes a summand indexed by $z = s$ if and only if $s = s^*$. Since $P^\sigma_{y,s} = 1$ if $y \in \{1,s\}$ and $P^\sigma_{y,s} = 0$ otherwise, we conclude that

$$P^\sigma_{y,w} = P^\sigma_{y,s\ltimes w} + q^2 P^\sigma_{s\ltimes y,s\ltimes w} + \delta'' \cdot q(P^\sigma_{y,s\ltimes w} - P^\sigma_{s\ltimes y,s\ltimes w}) - \delta \cdot q^2 P^\sigma_{y,rs\ltimes w} - \delta''' \cdot q$$

where $\delta''' = 1$ if $y = 1$ and $s = s^*$ and $w = srs$ and $\delta''' = 0$ otherwise. Note that if $\delta = 1$ then $P^\sigma_{y,rs\ltimes w} = P^\sigma_{s\ltimes y,rs\ltimes w}$ by Corollary 2.7. Thus to finish our proof, it is enough to check that

$$\delta''(P^\sigma_{y,s\ltimes w} - P^\sigma_{s\ltimes y,s\ltimes w}) - \delta''' = \delta'(P^\sigma_{1,s\ltimes w} - P^\sigma_{s,s\ltimes w})$$

This is clear if $y = 1$ and $s = s^*$ and $w \neq srs$ since then $\delta' = \delta'' = 1$ and $\delta''' = 0$. On the other hand, if $y = 1$ and $s = s^*$ but $w = srs$ then $\delta' = 0$ and $\delta'' = \delta''' = 1$ and $P^\sigma_{y,s\ltimes w} - P^\sigma_{s\ltimes y,s\ltimes w} = P^\sigma_{1,r} = 1$, which again gives equality. Finally, if $y \neq 1$ or $s \neq s^*$ then $\delta' = \delta'' = \delta''' = 0$ and our equation again holds. $\square$

3.3 Four technical propositions

To prove Conjectures A′ and B′ for the universal Coxeter system (W, S) we require an analog of Proposition 3.4. The requisite statement splits into four somewhat more technical propositions, which we prove here. The Coxeter system (W, S) is always assumed to be universal in this section (and we stop stating this condition in our results.)

Mirroring the notation $P_{y,z;w}$ from (3.1), given $y, z, w \in \mathbf{I}_*$ with $y \leq z$, we define

$$P^\sigma_{y,z;w} = P^\sigma_{y,w} - P^\sigma_{z,w}. \tag{3.5}$$

Also, given elements $w_1, w_2, \dots, w_k \in W$ we write $\langle w_1, w_2, \dots, w_k\rangle$ for the subgroup they generate. Finally, recall from Theorem-Definition 2.3 that we denote the rank function on $(\mathbf{I}_*, \leq)$ by

$$\rho : \mathbf{I}_* \to \mathbb{N},$$

so that $\rho(w)$ is the length of any reduced $\mathbf{I}_*$-expression for $w \in \mathbf{I}_*$.

At least one half of the following result is well-known, being equivalent to the fact that the Kazhdan-Lusztig polynomials of dihedral Coxeter systems are all constant.

Proposition 3.8. Let $y, z, w \in \mathbf{I}_*$ with $y \leq z$. If $r, s \in S$ such that $w \in \langle r, s\rangle$, then

$$P^\sigma_{y,z;w} = P_{y,z;w} = \begin{cases} 1 & \text{if } y \leq w \text{ and } z \not\leq w \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If suffices to show that $P^\sigma_{y,w} = P_{y,w} = 1$ if $y \leq w$; however, this follows by a straightforward argument using induction on the length of w and Lemmas 3.3 and 3.7. In particular, the base cases for our induction are given by Corollaries 2.7 and 2.12, which show that $P^\sigma_{y,w} = 1$ if $y \leq w$ and $\rho(w) \leq 1$, and that $P_{y,w} = 1$ if $y \leq w$ and $\ell(w) \leq 1$. □

For the duration of this section we adopt the following specific setup: fix $y, z \in \mathbf{I}_*$ with $y \leq z$ and assume $w \in \mathbf{I}_*$ has the form

$$w = \underbrace{srsrs\cdots}_{k+1 \text{ factors}} \ltimes u \tag{3.6}$$

where

- k is a positive integer;
- $r, s \in S$ such that $r \neq s$ and $s \notin \mathrm{Des}_L(y)$ and $s \notin \mathrm{Des}_L(z)$;
- $u \in \mathbf{I}_*$ such that $\{r, s\} \cap \mathrm{Des}_L(u) = \varnothing$.

In addition, define $a \in \langle r, s\rangle \subset W$ as the element

$$a = \underbrace{\cdots srsrs}_{k \text{ factors}} \tag{3.7}$$

and let $y', z', w' \in \mathbf{I}_*$ denote the twisted involutions

$$y' = a \ltimes y \qquad \text{and} \qquad z' = a \ltimes z \qquad \text{and} \qquad w' = a \ltimes w. \tag{3.8}$$

Observe that $\rho(y') = \rho(a) + \rho(y)$ and $\rho(z') = \rho(a) + \rho(z)$ and $\rho(w') = \rho(u) + 1$, and that clearly $y' \leq z'$ in the Bruhat order. In addition, w' is given by either $s \ltimes u$ or $r \ltimes u$, depending on the parity of k. We now have our second proposition.

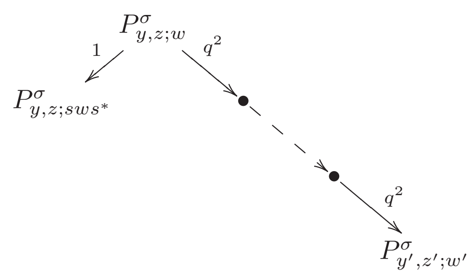

Figure 1: Labelled tree illustrating part (a) Proposition 3.9.

Proposition 3.9. Suppose $w \notin \langle r, s\rangle$ and either $y \neq 1$ or $s \neq s^*$. Then

(a) $P^{\sigma}_{y,z;w} = P^{\sigma}_{y,z;sws^*} + q^{2k} P^{\sigma}_{y',z';w'}$.

(b) $P_{y,z;w} = P_{y,z;sws^*} + q^{2k} P_{y',z';w'} + 2q^k P_{ay,az;aws^*}$.

Remark. The best way of making sense of this and the next two propositions is through pictures. The recurrences in each proposition are conveniently illustrated as trees whose nodes are labelled by the polynomials $P^{\sigma}_{y,z;w}$ or $P_{y,z;w}$ and whose edges are labelled by powers of q; see Figures 1, 2, 3, 4, 5, and 6. In these diagrams, the branches at each level indicate one application of Lemma 3.7 or Lemma 3.3; these lemmas add two or three children to a given node while possibly also canceling a node two levels down the tree. This cancelation accounts for the chains of k single-child nodes, which appear as dashed lines.

Proof. First consider Figure 1. The proof of part (a) is very similar to that of Proposition 3.4, but using Lemma 3.3 in place of Lemma 3.7. The argument is entirely analogous because, under our current hypotheses, whenever we apply Lemma 3.7 the second indicator δ' defined in that result is zero.

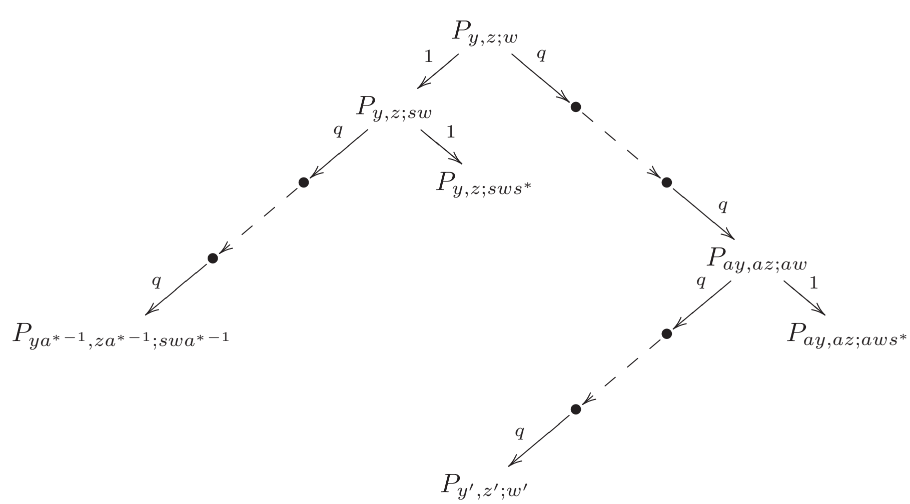

Figure 2: Labelled tree illustrating part (b) Proposition 3.9.

Now consider Figure 2. To prove part (b), we first apply the right-handed version of Proposition 3.4 to $P_{y,z;w}$ and then apply the left-handed version of Proposition 3.4 to the result. In detail, the first application gives

$$P_{y,z;w} = P_{y,z;ws^*} + q^k P_{ya^{*-1},za^{*-1};wa^{*-1}}$$

while the second gives $P_{y,z;ws^*} = P_{y,z;sws^*} + q^k P_{ay,az;aws^*}$ and

$$P_{ya^{*-1},za^{*-1};wa^{*-1}} = P_{ya^{*-1},za^{*-1};swa^{*-1}} + q^k P_{y',z';w'}.$$

Since $P_{ay,az;aws^*} = P_{ya^{*-1},za^{*-1};swa^{*-1}}$ by Corollary 2.12 combining the preceding equations gives the desired recurrence. ☐

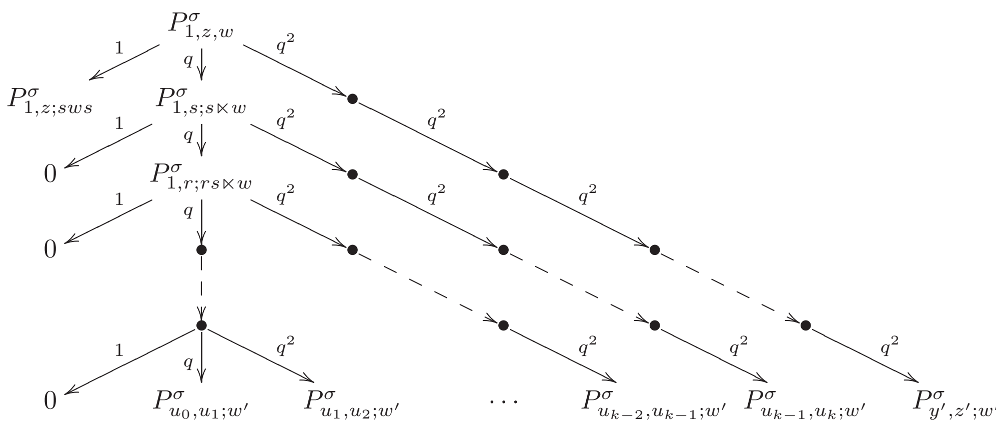

Figure 3: Labelled tree illustrating part (a) of Proposition 3.10

We proceed immediately to our next proposition.

Proposition 3.10. Suppose $w \notin \langle r, s\rangle$ and $y = 1 \neq z$ and $s = s^*$ and $r = r^*$. Then there are elements $u_0, u_1, \ldots, u_k \in \mathbf{I}_*$ and $z_1, z_2, \ldots, z_k \in W$ with $u_i \leq u_{i+1}$ and $u_i \leq z_i$ such that

(a) $P^\sigma_{y,z;w} = P^\sigma_{y,z;sws} + q^{2k} P^\sigma_{y',z';w'} + \sum_{0\leq i<k} q^{i+k} P^\sigma_{u_i,u_{i+1};w'}$.

(b) $P_{y,z;w} = P_{y,z;sws} + q^{2k} P_{y',z';w'} + \sum_{0\leq i<k} q^{i+k} \left(P_{u_i,u_{i+1};w'} + 2P_{u_{i+1},z_{i+1};w'}\right)$.

Proof. The twisted involutions $u_0, u_1, \ldots, u_k \in \mathbf{I}_*$ are defined as follows:

- If $k-i$ is even then let $u_i = (\cdots srsrs) \ltimes 1$, where $(\cdots srsrs)$ has i factors.
- If $k-i$ is odd then let $u_i = (\cdots rsrsr) \ltimes 1$, where $(\cdots rsrsr)$ has i factors.

Consider Figure 3. To prove part (a), we note that Lemma 3.7 implies

$$P^\sigma_{y,z;w} = P^\sigma_{1,z;s\ltimes w} + q^2(P^\sigma_{s,s\ltimes z;s\ltimes w} - \delta \cdot P^\sigma_{s,s\ltimes z;rs\ltimes w}) + qP^\sigma_{1,s;s\ltimes w}$$

where $\delta = 0$ if $k = 1$ and $\delta = 1$ otherwise. If $\delta = 1$ then Proposition 3.9 gives $P^\sigma_{s,s\ltimes z;s\ltimes w} = P^\sigma_{s,s\ltimes z;rs\ltimes w} + q^{2(k-1)} P^\sigma_{y',z';w'}$,; by substituting this into the previous equation we get in either case

$$P^\sigma_{y,z;w} = P^\sigma_{1,z;s\ltimes w} + q^{2k} P^\sigma_{y',z';w'} + qP^\sigma_{1,s;s\ltimes w}. \tag{3.9}$$

If $k = 1$ then this equation coincides with the recurrence in part (a), and if $k > 1$ then by induction (with the parameters (k, r, s, y, z, w) replaced by $(k-1, s, r, 1, s, s \ltimes w)$) we may assume that

$$P^\sigma_{1,s;s\ltimes w} = P^\sigma_{1,s;rs\ltimes w} + q^{2(k-1)} P^\sigma_{u_{k-1},u_k;w'} + \sum_{0 \le i < k-1} q^{i+k-1} P^\sigma_{u_i,u_{i+1};w'}.$$

Since here $P^\sigma_{1,s;rs\ltimes w} = P^\sigma_{1,1;rs\ltimes w} = 0$ as $s \in \mathrm{Des}_L(rs \ltimes w)$, substituting the previous equation into (3.9) establishes part (a) for all k.

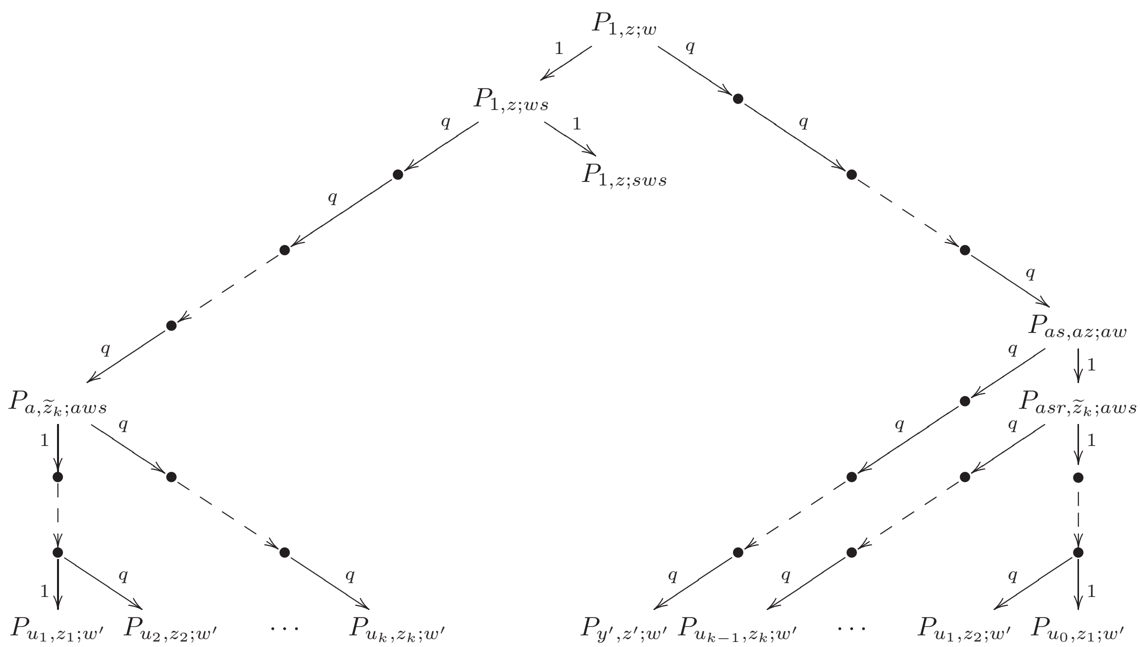

Figure 4: Labelled tree illustrating part (b) of Proposition 3.10

Before proving part (b) we must define the elements $z_i \in W$. For this, we first define an intermediate sequence $\widetilde{z}_1, \widetilde{z}_2, \dots, \widetilde{z}_{k+1} \in W$ in the following way. Set $\widetilde{z}_{k+1} = az$ where a is given by (3.7), and for $i \le k$ define $\widetilde{z}_i$ inductively by these cases:

- If $k - i$ is even then let $\widetilde{z}_i$ be the element with smaller length in the set $\{\widetilde{z}_{i+1}, \widetilde{z}_{i+1}r^*\}$.
- If $k - i$ is odd then let $\widetilde{z}_i$ be the element with smaller length in the set $\{\widetilde{z}_{i+1}, \widetilde{z}_{i+1}s^*\}$.

Note by construction that $\widetilde{z}_i r^* > \widetilde{z}_i$ if $k - i$ is even and $\widetilde{z}_i s^* > \widetilde{z}_i$ if $k - i$ is odd. Finally, define $z_1, z_2, \dots, z_k \in W$ as follows:

- If $k - i$ is even then let $z_i = \widetilde{z}_i(rsrsr\cdots)^*$ where $(rsrsr\cdots)$ has $i - 1$ factors.
- If $k - i$ is odd then let $z_i = \widetilde{z}_i(srsrs\cdots)^*$ where $(srsrs\cdots)$ has $i - 1$ factors.

Note that by construction $\ell(z_i) = \ell(\widetilde{z}_i) + i - 1$. Note also that since we assume $s = s^*$ and $r = r^*$, the $*$'s in the preceding bullet points are superfluous; however, these will be significant in the proof of the next proposition when we refer to the definition of u_{k-1}, u_k, and z_k.

Now consider Figure 4. To prove part (b), we note from Proposition 3.4 that

$$P_{y,z;w} = P_{1,z;sw} + q^k P_{a,az;aw} = P_{1,z;ws} + q^k P_{as,az;aw}.$$

Here the second equality follows from properties in Corollaries 2.12 (in particular, the fact that $P_{y,w} = P_{ys,w}$ if $ws < w$). One checks similarly that applying (the left- and right-handed versions of) Proposition 3.4 to the terms on the right gives

$$P_{y,z;w} = P_{1,z;sws} + q^k P_{a,\widetilde{z}_k;aws} + q^k P_{asr,\widetilde{z}_k;aws} + q^{2k} P_{y',z';w'}. \tag{3.10}$$

From here, it is a straightforward exercise to check the identities

$$P_{a,\widetilde{z}_k;aws} = \sum_{i=0}^{k-1} q^i P_{u_{i+1},z_{i+1};w'} \qquad \text{and} \qquad P_{asr,\widetilde{z}_k;aws} = \sum_{i=0}^{k-1} q^i P_{u_i,z_{i+1};w'}$$

which on substitution afford the desired recurrence (since $P_{u_{i-1},z_i;w'} + P_{u_i,z_i;w'} = P_{u_{i-1},u_i;w'} + 2P_{u_i,z_i;w'}$). In particular, one obtains these identities by applying the right-handed version Proposition 3.4 to the left hand sides, and then applying the proposition again to the term in the result with coefficient one, repeating this process until the third index of every polynomial is w'. ☐

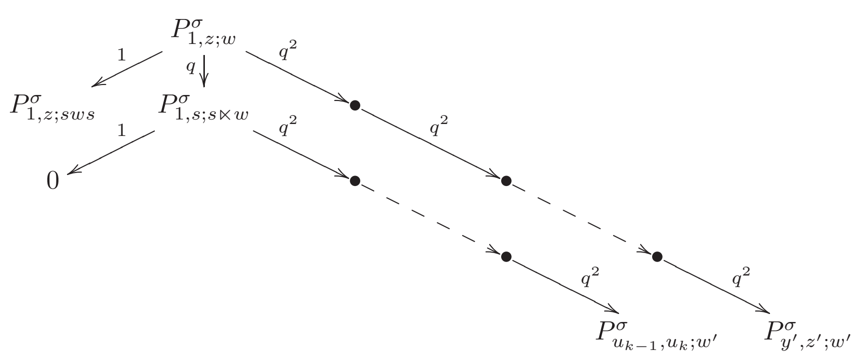

Figure 5: Labelled tree illustrating part (a) of Proposition 3.11

For this section's final proposition, it is convenient to let $y'', z'', w'' \in W$ denote the elements

$$y'' = a \qquad \text{and} \qquad z'' = \begin{cases} azr^* & \text{if } r^* \in \mathrm{Des}_R(z) \\ az & \text{otherwise} \end{cases} \qquad \text{and} \qquad w'' = awsr^*. \tag{3.11}$$

We remark that in the notation of the proof of the previous proposition, the element $z'' = \widetilde{z}_k$. Thus we also have $z_k = z''(rsrsr\cdots)^*$ where $(rsrsr\cdots)$ has $k-1$ factors.

Proposition 3.11. Suppose $y = 1 \neq z$ and $s = s^*$ and $r \neq r^*$ (so that automatically $w \notin \langle r, s\rangle$). Then, with $u_{k-1}, u_k \in \mathbf{I}_*$ and $z_k \in W$ defined as in the proof of Proposition 3.10, we have

(a) $P^\sigma_{y,z;w} = P^\sigma_{y,z;sws} + q^{2k} P^\sigma_{y',z';w'} + q^{2k-1} P^\sigma_{u_{k-1},u_k;w'}$.

(b) $P_{y,z;w} = P_{y,z;sws} + q^{2k} P_{y',z';w'} + q^{2k-1}\left(P_{u_{k-1},u_k;w'} + 2P_{u_k,z_k;w'}\right) + \begin{cases} 2q^k P_{y'',z'';w''} & \text{if } k > 1 \\ 0 & \text{if } k = 1. \end{cases}$

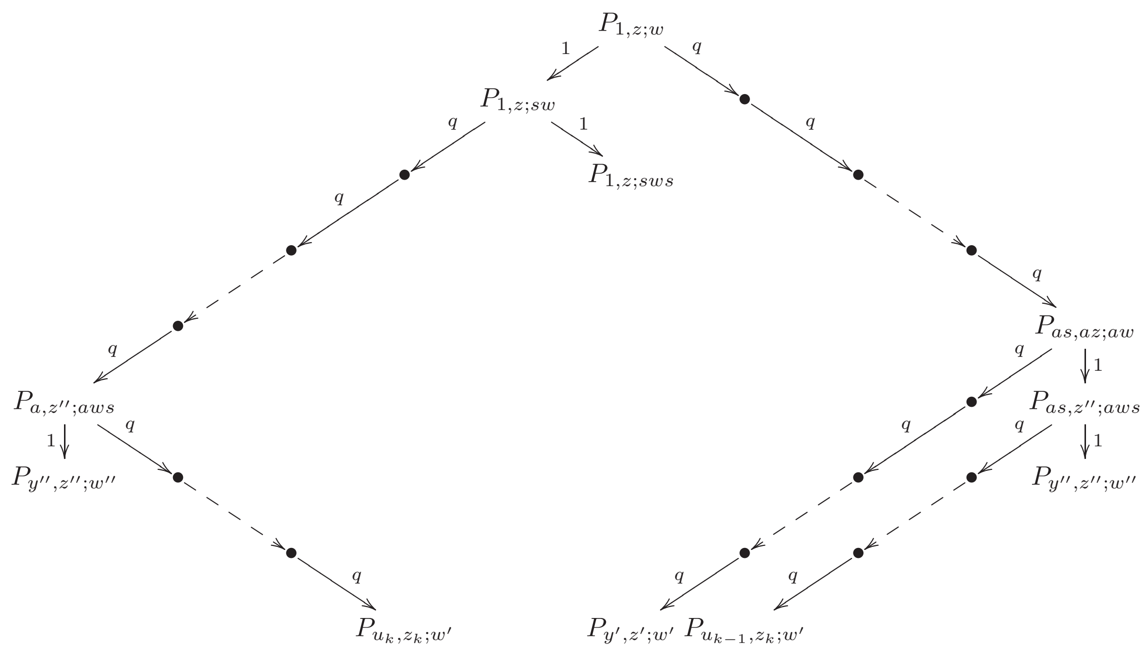

Figure 6: Labelled tree illustrating part (b) of Proposition 3.11 (when $k > 1$)

Proof. Consider Figure 5. To prove part (a), we first note that the argument used to show (3.9) in the previous proposition remains valid here and gives

$$P^{\sigma}_{y,z;w} = P^{\sigma}_{1,z;s\ltimes w} + q^{2k} P^{\sigma}_{y',z';w'} + q P^{\sigma}_{1,s;s\ltimes w}.$$

If $k = 1$ then (using the definitions in the proof of Proposition 3.10) we have $u_0 = 1$ and $u_1 = s$ and so this equation coincides with the desired recurrence. If $k > 1$, then since $r \neq r^*$, we can apply Proposition 3.8 with the parameters (k, r, s, y, z, w) replaced by $(k - 1, s, r, 1, s, s \ltimes w)$ to obtain

$$P^{\sigma}_{1,s;s\ltimes w} = P^{\sigma}_{1,s;rs\ltimes w} + q^{2(k-1)} P^{\sigma}_{u_{k-1},u_k;w'}.$$

Here $u_{k-1} = (\cdots rsrsr) \ltimes 1$ where $(\cdots rsrsr)$ has $k - 1$ factors and $u_k = (\cdots srsrs) \ltimes 1$ where $(\cdots srsrs)$ has k factors. Substituting this identity into our formula for $P^{\sigma}_{y,z,w}$ then establishes part (a) for all k.

To prove part (b), consider Figure 6 and observe that it follows by successive applications of Propositions 3.4, exactly as in the proof of Proposition 3.10, that

$$P_{y,z;w} = P_{1,z;sws} + q^k P_{a,z'';aws} + q^k P_{as,z'';aws} + q^{2k} P_{y',z';w'}.$$

Note that the third term on the right $q^k P_{as,z'';aws}$ differs from the analogous equation (3.10) above; this is because now we have $asr^* \not< as$ since $r \neq r^*$.

Now, if $k = 1$ then $u_{k-1} = u_0 = as = 1$ and $u_k = u_1 = a = s$ and $z_k = z_1 = z''$ and $w' = aws$, so the preceding formula for $P_{y,z;w}$ coincides with the desired recurrence as $P_{u_{k-1},z_k;w'} + P_{u_k,z_k;w'} =$

$P_{u_{k-1},u_k;w'} + 2P_{u_k,z_k;w'}$. Alternatively, if $k > 1$ then the right-handed version of Proposition 3.4 with the parameters (k, r, s, y, z, w) replaced by $(k-1, s, r, -, -, aws)$ gives

$$P_{a,z'';aws} = P_{y'',z'';w''} + q^{k-1}P_{u_k,z_k;w''} \qquad \text{and} \qquad P_{as,z'';aws} = P_{as,z'';w''} + q^{k-1}P_{u_{k-1},z_k;w''}.$$

Since $w''s < w''$ as $k > 1$, we have $P_{as,z'';w''} = P_{a,z'';w''} = P_{y'',z'';w''}$, and so substituting these two identities into our previous equation gives the desired recurrence in all cases. $\square$

Our first application of these results is the following theorem, which shows that the perhaps most natural analogues of Conjectures A and B for twisted involutions (which are false in general) do hold in the universal case.

Theorem 3.12. If (W, S) is a universal Coxeter system, then the difference $P^\sigma_{y,w} - P^\sigma_{z,w}$ has nonnegative integer coefficients for all $y, z, w \in \mathbf{I}_*$ with $y \leq z$ in the Bruhat order. In particular, we have $P^\sigma_{y,w} \in \mathbb{N}[q]$ for each $y, w \in \mathbf{I}_*$.

Proof. The proof is by induction on $\rho(w)$, and is similar to that of Theorem 3.5. Fix $y, z, w \in \mathbf{I}_*$ with $y < z$. If $\rho(w) \leq 1$ then the theorem follows from Proposition 3.8. Suppose $\rho(w) \geq 2$, and that $s \in \mathrm{Des}_L(w)$. By Corollary 2.7 we may assume that $s \notin \mathrm{Des}_L(y)$ and $s \notin \mathrm{Des}_L(z)$, in which case one checks that the triple (y, z, w) satisfies the hypotheses of one of Propositions 3.8, 3.9, 3.10, or 3.11. These propositions then imply $P^\sigma_{y,z;w} \in \mathbb{N}[q]$ by induction. $\square$

Recall from Proposition 2.13 that the polynomials

$$P^\pm_{y,w} = \tfrac{1}{2}\left(P_{y,w} \pm P^\sigma_{y,w}\right)$$

have integer coefficients for all $y, w \in \mathbf{I}_*$. Conjecture A$'$ asserts that these coefficients are nonnegative, while Conjecture B$'$ asserts the stronger property that the polynomials $P^\pm_{y,w}$ are decreasing with respect to the index y (and the Bruhat order). As the main result of this section we now prove that these conjectures hold for universal Coxeter systems.

Theorem 3.13. If (W, S) is a universal Coxeter system then the polynomials $P^+_{y,w} - P^+_{z,w}$ and $P^-_{y,w} - P^-_{z,w}$ have nonnegative integer coefficients for all $y, z, w \in \mathbf{I}_*$ with $y \leq w$ in the Bruhat order. In particular, we have $P^+_{y,w} \in \mathbb{N}[q]$ and $P^-_{y,w} \in \mathbb{N}[q]$ for each $y, w \in \mathbf{I}_*$.

Proof. Recall that the coefficients of $P_{y,z;w} \pm P^\sigma_{y,z;w}$ are all even by Proposition 2.13. Since $P_{y,z;w}$ and $P^\sigma_{y,z;w}$ both have positive coefficients by Theorems 3.5 and 3.12, it suffices just to show that $P_{y,z;w} - P^\sigma_{y,z;w} \in \mathbb{N}[q]$ for $y, z, w \in \mathbf{I}_*$ with $y < z$. One can prove this fact by induction on $\rho(w)$ using the same argument as in the proof of Theorem 3.12. The same inductive argument works because the differences between parts (a) and (b) in each of our propositions in this section involves only polynomials $P_{y,z;w}$ and differences $P_{y,z;w} - P^\sigma_{y,z;w}$. $\square$

3.4 Structure constants

In the rest of this paper, we redirect our focus to Conjecture C$'$. Continue to assume (W, S) is a universal Coxeter system. This section describes an inductive method of computing the Laurent polynomials $(h_{x,y;z})_{x,y,z\in W}$ and $\left(h^\sigma_{x,y;z}\right)_{x\in W,\, y,z\in \mathbf{I}_*}$, which we recall are the structure constants in $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ satisfying

$$c_x c_y = \sum_{z\in W} h_{x,y;z} c_z \qquad \text{and} \qquad C_x A_y = \sum_{z\in \mathbf{I}_*} h^\sigma_{x,y;z} A_z.$$

Note here that we write $(c_w)_{w\in W}$ for the Kahzdan-Lusztig basis of the Hecke algebra $\mathcal{H}_q$ of W, and $(C_w)_{w\in W}$ for the Kazhdan-Lusztig basis of the Hecke algebra $\mathcal{H}_{q^2}$ with parameter q^2. In turn, recall that $(A_w)_{w\in \mathbf{I}_*}$ denotes the distinguished basis of $\mathcal{M}_{q^2}$ given by Theorem-Definition (1.3).

We begin by recollecting the results of Dyer [4] concerning the Kazhdan-Lusztig basis for universal Coxeter systems. The following appears as [4, Definition 3.11].

Definition 3.14. Assume (W,S) is a universal Coxeter system. Let $w\in W$ and $n=\ell(w)$, and suppose $s_i\in S$ such that $w=s_1s_2\cdots s_n$. For each integer $j\in\mathbb{Z}$, define $c(w,j)\in\mathcal{H}_q$ recursively according to the following cases:

(a) If $2\leq j\leq n-1$ (so that $n\geq 3$) and $s_{j-1}=s_{j+1}$, then set

$$c(w,j)=c_{w'}+c(w',j-1), \qquad \text{where } w'=s_1\cdots\widehat{s_j}\widehat{s}_{j+1}\cdots s_n.$$

Here, we write $\widehat{s_j}$ to indicate that the factor s_j is omitted.

(b) Otherwise set $c(w,j)=0$.

The following result of Dyer [4, Theorem 3.12] gives the decomposition of the product c_xc_y in terms of the Kazhdan-Lusztig basis of $\mathcal{H}_q$, and shows that the Laurent polynomials $(h_{x,y;z})_{x,y,z\in W}$ have nonnegative coefficients, and are in fact polynomials in $v+v^{-1}$ with nonnegative integer coefficients. (This latter property does not hold in general.)

Theorem 3.15 (Dyer [4])**.** Assume (W,S) is a universal Coxeter system. Let $x,y\in W$ and $n=\ell(x)$. Then

$$c_xc_y=\begin{cases}(v+v^{-1})\,(c_{xsy}+c(xsy,n)) & \text{if } \mathrm{Des}_R(x)=\mathrm{Des}_L(y)=\{s\}\neq\varnothing\\ c_{xy}+c(xy,n)+c(xy,n+1) & \text{otherwise.}\end{cases}$$

Remark. The preceding theorem differs from the corresponding statement in [4] as a result of our notational conventions. In [4, Theorem 3.12], Dyer writes "C_w" to denote the element of $\mathcal{H}_q$ which in our notation is written

$$\sum_{y\in W}(-v)^{\ell(w)-\ell(y)}\cdot P_{y,w}(q^{-1})\cdot v^{-\ell(y)}\cdot t_y.$$

This element is just $\mathrm{sgn}(w)\cdot\iota(c_w)$, where ι is the $\mathcal{A}$-algebra automorphism of $\mathcal{H}_q$ with $t_w\mapsto(-q)^{\ell(w)}\cdot t_{w^{-1}}^{-1}$ for $w\in W$. (When checking this it helps to recall $\overline{c_w}=c_w$.) This observation transforms Dyer's results into what is stated here.

Moving on to the analogous decomposition of C_xA_y, we have this lemma. Recall from Theorem 2.6 that if $s\in S$ then $C_s=q^{-1}(T_s+1)\in\mathcal{H}_{q^2}$.

Lemma 3.16. Assume (W,S) is a universal Coxeter system. Suppose $s\in S$ and $w\in\mathbf{I}_*$.

(a) If $s\in\mathrm{Des}_L(w)$ then $C_sA_w=\left(q+q^{-1}\right)A_w$.

(b) If $s \notin \mathrm{Des}_L(w)$ then

$$C_s A_w = \begin{cases} A_{sws^*} + A_{rwr^*} & \text{if } \mathrm{Des}_L(w) = \{r\} \text{ and } \mathrm{Des}_L(rwr^*) = \{s\} \\ A_{sws^*} + A_s & \text{if } w \in S \text{ and } s = s^* \\ (v+v^{-1})A_s & \text{if } w = 1 \text{ and } s = s^* \\ A_{sws^*} & \text{otherwise.} \end{cases}$$

Proof. Part (a) is immediate from Theorem 2.6. If $w = 1$ then $m^\sigma(y \xrightarrow{s} 1) =$ for all $y \in \mathbf{I}_*$ with $sy < y$ so by Theorem 2.6 we have $C_s A_1 = (v+v^{-1})^c A_{s \ltimes 1}$ where $c = \delta_{s,s^*}$. This proves part (b) when $w = 1$.

For the remaining cases, assume $w \neq 1$ and $\mathrm{Des}_L(w) = \{r\} \neq \{s\}$. Combining Theorem 2.6 and Lemma 3.6 gives $C_s A_w = A_{s \ltimes 1} + \sum_{y \in X} A_y$, where $X \subset \mathbf{I}_*$ is the subset which contains s if $s = s^*$ and $w = r \in S$, and which contains rwr^* if $rwr^* \in \mathbf{I}_*$ and $\mathrm{Des}_L(rwr^*) = \{s\}$. Since rwr^* actually always belongs to $\mathbf{I}_*$ and since $\mathrm{Des}_L(rwr^*) = \{s\}$ implies $w \notin S$, the set X contains at most one element and our formula $C_s A_w = A_{s \ltimes 1} + \sum_{y \in X} A_y$ reduces to the cases in the lemma. $\square$

We now make this definition, after Definition 3.14.

Definition 3.17. Assume (W,S) is a universal Coxeter system. Let $w \in \mathbf{I}_*$ and $n = \rho(w)$, and suppose $s_i \in S$ such that $w = s_1 \ltimes s_2 \ltimes \cdots \ltimes s_n \ltimes 1$. For each integer $j \in \mathbb{Z}$, we define $A(w,j) \in \mathcal{M}_{q^2}$ recursively according to the following cases:

(a) If $2 \leq j \leq n-1$ (so that $n \geq 3$) and $s_{j-1} = s_{j+1}$, then we set

$$A(w,j) = A_{w'} + A(w', j-1), \qquad \text{where } w' = s_1 \ltimes \cdots \ltimes \widehat{s}_j \ltimes \widehat{s}_{j+1} \ltimes \cdots \ltimes s_n.$$

Here, we again write $\widehat{s}_j$ to indicate that the factor s_j is omitted.

(b) If $j = n$ and $n \geq 2$ and $\{s_{n-1}, s_n\} \subset \mathbf{I}_*$, then we set

$$A(w,j) = A_{w'} + A(w', n-1), \qquad \text{where } w' = s_1 \ltimes \cdots \ltimes s_{n-1}.$$

(c) Otherwise we set $A(w,j) = 0$.

Using this notation, the following analog of Theorem 3.15 now decomposes the product $C_x A_y$ in terms of the distinguished basis $(A_z)_{z \in \mathbf{I}_*}$ of $\mathcal{M}_{q^2}$. This result shows that the Laurent polynomials $(h_{x,y;z})_{x,y,z \in W}$ have nonnegative coefficients, but in contrast to our previous situation, $h_{x,y;z}$ does not typically have nonnegative coefficients when written as a polynomial in $v + v^{-1}$.

Theorem 3.18. Assume (W,S) is a universal Coxeter system. If $x \in W$ and $y \in \mathbf{I}_*$ and $n = \ell(x)$, then

$$C_x A_y = \begin{cases} (v+v^{-1})\left(A_{x \ltimes 1} + A(x \ltimes 1, n)\right) & \text{if } x \neq 1 \text{ and } y = 1 \text{ and } \mathrm{Des}_R(x) \subset \mathbf{I}_* \\ (q+q^{-1})\left(A_{xs \ltimes y} + A(xs \ltimes y, n)\right) & \text{if } \mathrm{Des}_R(x) = \mathrm{Des}_L(y) = \{s\} \neq \varnothing \\ A_{x \ltimes y} + A(x \ltimes y, n) + A(x \ltimes y, n+1) & \text{otherwise.} \end{cases}$$

Proof. The proof is similar to that of [4, Theorem 3.12], and proceeds by induction on n. If $n \leq 1$ then the theorem reduces to Lemma 3.16 (checking this fact is a healthy exercise which we leave to the reader), so we may assume $\ell(x) \geq 2$ and that

$$x = x'rs \qquad \text{for some } x' \in W \text{ and } r, s \in S \text{ with } \ell(x') = \ell(x) - 2.$$

It follows from Theorem 3.15 (noting that the $\mathbb{Z}$-linear map $\mathcal{H}_q \to \mathcal{H}_{q^2}$ with $v^n \mapsto q^n$ and $t_w \mapsto T_w$ is a ring embedding with $c_w \mapsto C_w$) that

$$C_x = \begin{cases} C_{xs}C_s - C_{x'} & \text{if } \mathrm{Des}_R(x') = \{s\} \\ C_{xs}C_s & \text{otherwise.} \end{cases} \tag{3.12}$$

It suffices to consider the following five cases:

(i) Suppose $y = 1$. Then $A(x \ltimes y, n+1) = 0$ and so we wish to show that $C_x A_1 = (v + v^{-1})^c \cdot (A_{v \ltimes 1} + A(v \ltimes 1, n))$ where $c = |\{s\} \cap \mathbf{I}_*|$.

(ii) Suppose $s \in \mathrm{Des}_L(y)$. We then wish to show that $C_x A_y = (q + q^{-1}) \left(A_{xs \ltimes y} + A(xs \ltimes y, n)\right).$

(iii) Suppose $y \in S$ and $s \notin \mathrm{Des}_L(y)$ and $s = s^*$. Then $A(x \ltimes y, n+1) = A_{x \ltimes 1} + A(x \ltimes 1, n)$ and so we wish to show $C_x A_y = A_{x \ltimes y} + A(x \ltimes y, n) + A_{x \ltimes 1} + A(x \ltimes 1, n)$.

(iv) Suppose $\rho(y) = 1$ and $s \notin \mathrm{Des}_L(y)$ but either $y \notin S$ or $s \neq s^*$. Then $A(x \ltimes y, n+1) = 0$ and so we wish to show $C_x A_y = A_{x \ltimes y} + A(x \ltimes y, n)$.

(v) Suppose $\rho(y) \geq 2$ and $s \notin \mathrm{Des}_L(y)$. We then wish to show that $C_x A_y = A_{x \ltimes y} + A(x \ltimes y, n) + A(x \ltimes y, n+1)$.

The proof of each case is similar, and involves substituting (3.12) for C_x and then applying Lemma 3.16 and induction. Case (v) is the most complicated, but its proof is nearly the same as that of [4, Lemma 6.2]. We demonstrate (i) as an example and leave the rest to the reader.

For case (i), suppose $y = 1$ and let $c = |\{s\} \cap \mathbf{I}_*|$; recall that $\mathrm{Des}_R(x) = \{s\}$ by assumption. If $\mathrm{Des}_R(x') \neq \{s\}$ then $C_x = C_{x'r}C_s$ by (3.12) and $A(x \ltimes 1, n-1) = 0$, in which case by Lemma 3.16 and then induction we get

$$\begin{aligned} C_x A_1 &= C_{x'r} C_s A_1 \\ &= (v + v^{-1})^c \cdot C_{x'r} A_{s \ltimes 1} \\ &= (v + v^{-1})^c \cdot (A_{x \ltimes 1} + \underbrace{A(x \ltimes 1, n-1)}_{=0} + A(x \ltimes 1, n)), \end{aligned}$$

which is what we want to show. Alternatively, if $\mathrm{Des}_R(x') = \{s\}$ then $C_x = C_{x'r}C_s - C_{x'}$ by (3.12) and $A(x \ltimes 1, n-1) = A_{x' \ltimes 1} + A(x' \ltimes 1, n-2)$, so by induction $C_{x'} A_1 = (v + v^{-1})^c \cdot A(x \ltimes 1, n-1)$. In this case by Lemma 3.16 and then induction we have

$$\begin{aligned} C_x A_1 &= (C_{x'r} C_s - C_{x'}) A_1 \\ &= (v + v^{-1})^c \cdot C_{x'r} A_{s \ltimes 1} - C_{x'} A_1 \\ &= (v + v^{-1})^c \cdot (A_{x \ltimes 1} + A(x \ltimes 1, n)) + \underbrace{(v + v^{-1})^c \cdot A(x \ltimes 1, n-1) - C_{x'} A_1}_{=0} \end{aligned}$$

which is again the desired formula. ∎

Wrapping up, we have this corollary immediately from Theorems 3.15 and 3.18.

Corollary 3.19. If (W,S) is a universal Coxeter system then each of the families

$$(h_{x,y;z})_{x,y,z\in W} \qquad \text{and} \qquad (\widetilde{h}_{x,y;z})_{x,y,z\in W} \qquad \text{and} \qquad (h^\sigma_{x,y;z})_{x\in W,\, y,z\in \mathbf{I}_*}$$

consists of Laurent polynomials in $\mathcal{A} = \mathbb{Z}[v,v^{-1}]$ with nonnegative coefficients.

3.5 Proof of the positivity conjecture for universal structure constants

As previously, (W,S) is a universal Coxeter system with a fixed S-preserving involution $* \in \mathrm{Aut}(W)$. Recall from (1.3) that for each $x\in W$ and $y,z\in \mathbf{I}_*$ we define two Laurent polynomials $h^\pm_{x,y;z}\in\mathcal{A}$ by the formula

$$h^\pm_{x,y;z} = \tfrac{1}{2}\left(\widetilde{h}_{x,y;z} \pm h^\sigma_{x,y;z}\right).$$

We devote this final section to proving Conjecture C′ for universal Coxeter systems—i.e., that every $h^\pm_{x,y;z}$ has nonnegative coefficients.

To begin, it is useful to recall the following notation from the proof of Proposition 2.14. Given $w\in W$, let $w^\dagger = w^{*-1}$ and more generally let

$$h\mapsto h^\dagger$$

denote the $\mathcal{A}$-linear map $\mathcal{H}_q\to\mathcal{H}_q$ with $(t_w)^\dagger = t_{w^\dagger}$ for $w\in W$. Observe that $\dagger$ is an anti-automorphism (of $\mathcal{A}$-algebras) and that $(c_w)^\dagger = c_{w^\dagger}$ for all $w\in W$ by Corollary 2.12. We now state two technical lemmas associated with Definitions 3.14 and 3.17.

Lemma 3.20. Assume (W,S) is a universal Coxeter system. Suppose $u,t\in W$ such that $\ell(u\ltimes t) = 2\ell(u)+\ell(t)$ and $\ell(t)\le 1$ and $t=t^*$. Fix an integer $n\le \ell(u)$. Then there exists a unique integer $k\ge 0$ and a unique sequence of elements

$$u = u_0 > u_1 > \cdots > u_k$$

in W such that $c(ut,n) = \sum_{i=1}^k c_{u_i t}$. This sequence has the following additional properties:

(a) For each $0\le i\le k$ we have $\ell(u_i\ltimes t) = 2\ell(u_i)+\ell(t)$.

(b) For any $w\in W$ with $\ell(utw) = \ell(u)+\ell(t)+\ell(w)$ we have

$$c(utw,n) = \sum_{i=1}^k c_{u_i tw} + c(u_k tw, n-k).$$

(c) Let $\delta = 1$ when $n-k = \ell(u_k)+1$ and let $\delta=0$ otherwise. Then we have

$$A(u\ltimes t,n) = \sum_{i=1}^k A_{u_i\ltimes t} + \delta\cdot A(u_k\ltimes t, n-k).$$

Remark. Note that we may have $k=0$ in this lemma; this indicates that $c(ut,n)=0$. In the case the sums $\sum_{k=1}^n$ are considered to be zero, and we automatically have $\delta = 0$ in part (c) since $n<\ell(u_0)+1$ by hypothesis.

Proof. We sketch the proof of this lemma, as everything derives from the definitions in a straightforward way by induction on $\ell(u)$. The existence of the sequence of elements $u = u_0 > u_1 > \cdots > u_k$, follows from Definition 3.14 by inspection, as does property (a). Property (b) holds because the first $k+1$ terms in the expansion of $c(utw, n)$ which one gets by applying Definition 3.14 successively depend only on the first $n+k$ factors in the unique reduced expression for utw. Part (c) follows from the fact that the same sequence of elements in S gives the unique reduced expression for ut and the unique reduced $\mathbf{I}_*$-expression for $u \ltimes t$. Noting this and comparing Definitions 3.14 and 3.17 (while remembering $n \leq \ell(u)$), we deduce that $A(u \ltimes t, n) = \sum_{i=1}^{k} A_{u_i \ltimes t} + A(u_k \ltimes t, n-k)$, and that $A(u_k \ltimes t, n-k)$ is zero unless $n - k = \rho(u_k \ltimes t)$. The latter condition is equivalent to having both $\ell(t) = 1$ and $n - k = \ell(u_k) + 1$; however, if $\ell(t) = 0$ while $n - k = \ell(u_k) + 1$ then $A(u_k \ltimes t, n-k)$ is zero by definition. $\square$

In the next lemma and theorem, we let $\Phi : \mathcal{M}_{q^2} \to \mathcal{H}_q$ denote the $\mathcal{A}$-linear map with $A_w \mapsto c_w$ for $w \in \mathbf{I}_*$.

Lemma 3.21. Assume (W, S) is a universal Coxeter system. Suppose $x \in W$ and $s \in S \cap \mathbf{I}_*$ such that $s \notin \mathrm{Des}_R(x)$. If $n = \ell(x)$, then

$$c(x \ltimes s, n+1) = \Phi\left(A(x \ltimes s, n+1)\right).$$

Proof. If $x = 1$ or if $\mathrm{Des}_L(x) \not\subset \mathbf{I}_*$ then the lemma holds since $c(x \ltimes s, n+1)$ and $A(x \ltimes s, n+1)$ are both zero. Assume $x \neq 1$ so that $x = x'r$ for some $y \in W$ and $r \in S \cap \mathbf{I}_*$ with $\ell(x') = \ell(x) - 1$. Then $c(x \ltimes s, n+1) = c_{x' \ltimes r} + c(x' \ltimes r, n)$ and $A(x' \ltimes s, n+1) = A_{x' \ltimes r} + A(x' \ltimes r, n)$, so the lemma follows by induction on n. $\square$

We may now state our final result, which establishes Conjecture C$'$ in the universal case.

Theorem 3.22. If (W, S) is a universal Coxeter system then the Laurent polynomials $h^{\pm}_{x,y,z}$ defined by (1.3) have nonnegative integer coefficients for all $x \in W$ and $y, z \in \mathbf{I}_*$.

Proof. Let $\mathcal{H}_q^+ = \mathbb{N}[v, v^{-1}]$-span$\{c_w : w \in W\}$ denote set of elements in $\mathcal{H}_q$ whose coefficients with respect to the Kazhdan-Lusztig basis $(c_w)_{w \in W}$ have nonnegative coefficients. Note that $\mathcal{H}_q^+$ is preserved by $\dagger$ since $(c_w)^\dagger = c_{w^\dagger}$.

Let $x \in W$ and $y \in \mathbf{I}_*$. By Theorems 3.15 and 3.18 we know that $c_x c_y c_{x^\dagger} \in \mathcal{H}_q^+$ and $\Phi(C_x A_y) \in \mathcal{H}_q^+$, and if we write $c_x c_y c_{x^\dagger} \pm \Phi(C_x A_y) = \sum_{z \in W} p_z^{\pm} c_z$ for some polynomials $p_z^{\pm} \in \mathbb{Z}[v, v^{-1}]$, then by definition $h^{\pm}_{x,y;z} = \frac{1}{2} p_z^{\pm}$ for each $z \in \mathbf{I}_*$. It is thus immediate that every $h^{+}_{x,y;z}$ has nonnegative coefficients, and to prove the theorem it is enough to show that

$$c_x c_y c_{x^{*-1}} - \Phi(C_x A_y) \in \mathcal{H}_q^+. \tag{3.13}$$

To this end, let $n = \ell(x)$. If $n = 0$ then (3.13) automatically holds since the left hand side is zero, so we may assume $n \geq 1$. We now consider the following cases in turn.

(a) Suppose $y = 1$. Expand the products $c_x c_y c_{x^\dagger} = c_x c_{x^\dagger}$ and $C_x A_y = C_x A_1$ according to Theorems 3.15 and 3.18. These expansions take one of two forms according to whether $s = s^*$, and applying Lemma 3.21 to the terms in either case shows that (3.13) holds.

(b) Suppose $y \neq 1$ and $\mathrm{Des}_L(y) \neq \mathrm{Des}_R(x)$. Apply Theorem 3.15 to expand the product $c_x c_y c_{x^\dagger}$, by expanding first $c_x c_y$ and then $(c_x c_y) c_{x^\dagger}$. There are again two cases according to whether $y \in S$. On comparing the resulting terms to Theorem 3.18 (while noting Lemma 3.21), one finds that (3.13) will hold if we can prove the following claims:

(i) If $\ell(y) \geq 1$ then we have $c(xy,n)c_{x^\dagger} - \Phi\left(A(x \ltimes y, n)\right) \in \mathcal{H}_q^+$.

(ii) If $\ell(y) \geq 2$ then we have $c(xy,n+1)c_{x^\dagger} - \Phi\left(A(x \ltimes y, n+1)\right) \in \mathcal{H}_q^+$.

To prove (i), write $y = ztz^\dagger$ where $z,t \in W$ such that $\ell(t) \leq 1$ and $t = t^*$ and $\ell(ztz^\dagger) = 2\ell(z) + \ell(t)$. Now let $u = xz$ and let $u = u_0 > u_1 > \cdots > u_k$ be the corresponding sequence of elements in W described in Lemma 3.20. Using part (b) of Lemma 3.20 and the fact that $\dagger$ is an anti-automorphism, we then have

$$\begin{aligned} c(xy,n)c_{x^\dagger} = c(utz^\dagger, n)c_{x^\dagger} &= \left(\sum_{i=1}^k c_{u_i tz^\dagger} + c(u_k tz^\dagger, n-k)\right) c_{x^\dagger} \\ &= \left(c_x \sum_{i=1}^k c_{zt(u_i)^\dagger}\right)^\dagger + c(u_k tz^\dagger, n-k)c_{x^\dagger} \\ &= \left(\sum_{i=1}^k c(ut(u_i)^\dagger, n)\right)^\dagger + \left(\text{an element of } \mathcal{H}_q^+\right). \end{aligned}$$

Here, the last equality follows by applying Theorem 3.15 to the terms in the sum on the second line. Since each $c(ut(u_i)^\dagger, n) = \sum_{j=1}^k c_{u_j t(u_i)^\dagger} + \left(\text{an element of } \mathcal{H}_q^+\right)$ by parts (a) and (b) of Lemma 3.20, after collecting terms in $\mathcal{H}_q^+$ we get

$$c(xy,n)c_{x^\dagger} = \sum_{i=1}^k c_{u_i \ltimes t} + c(u_k \ltimes t, n-k) + \left(\text{an element of } \mathcal{H}_q^+\right).$$

By part (c) of Lemma 3.20, however, we have $A(x \ltimes y, n) = \sum_{i=1}^k A_{u_i \ltimes t} + \delta \cdot A(u_k \ltimes t, n-k)$, where $\delta \in \{0,1\}$ is zero unless $n-k = \ell(x_k)+1$. If $\delta = 1$ then $A(u_k \ltimes t, n-k) = 0$ unless $t \in S \cap \mathbf{I}_*$, and so (i) follows by Lemma 3.21.

One proves (ii) by replacing n with $n+1$ in the preceding argument. Our applications of Lemma 3.20 remain valid after this substitution because we assume $\ell(y) \geq 2$, which implies $\ell(u) \geq 1$ and in turn $n+1 \leq \ell(u)$.

(c) Suppose $y \neq 1$ and $\mathrm{Des}_R(x) = \mathrm{Des}_L(y) = \{s\}$ for some $s \in S$. Again use Theorems 3.15 and 3.18 to expand the products $(c_x c_y)c_{x^\dagger}$ and $C_x A_y$. On comparing the resulting terms (while again noting Lemma 3.21) one finds that to prove (3.13) it is enough to show

$$(v+v^{-1}) \cdot c(xsy,n)c_{x^\dagger} = (q+q^{-1}) \cdot \Phi\left(A(xs \ltimes y, n)\right) + \left(\text{an element of } \mathcal{H}_q^+\right). \tag{3.14}$$

If $y = s \in \mathbf{I}_*$ then by Theorem 3.15 we have $c(xsy,n) = c_{xys} + \left(\text{an element of } \mathcal{H}_q^+\right)$ and in turn $c(xsy,n)c_{x^\dagger} = (v+v^{-1}) \cdot c(xs \ltimes y, n) + \left(\text{an element of } \mathcal{H}_q^+\right)$. Thus (3.14) holds in this case by Lemma 3.21.

If $\ell(y) \geq 2$ then the proof of (3.14) is similar to the arguments in part (b). A sketch goes as follows. First write $y = ztz^\dagger$ where $z,t \in W$ such that $\ell(t) \leq 1 \leq \ell(z)$ and $t^* = t$ and $\ell(ztz^\dagger) = 2\ell(z) + \ell(t)$. Let $u = xsz$ and let $u = u_0 > u_1 > \cdots > u_k$ be the corresponding sequence of elements in W. By now rewriting $c(xsy,n)$ in terms of the elements u_i and expanding various products using the properties in Lemma 3.20, one obtains

$$c(xsy,n)c_{y^\dagger} = (v+v^{-1})\left(\sum_{i=1}^k c_{u_i \ltimes t} + c(u_k \ltimes t, n-k)\right) + \left(\text{an element of } \mathcal{H}_q^+\right).$$

Comparing this to the formula for $A(xs \ltimes y, n)$ in part (c) of Lemma 3.20 then shows that (3.14) holds, as a consequence of Lemma 3.21.

□

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